CHARACTERISING ASPECTS OF PROOF COMPRESSION

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Abstract. We consider deep inference proof formalisms, which are flexible enough to embed many widely used proof systems, and construe compression mechanisms 'cut' and 'dag' in a way that is independent of any particular proof system, but in a way that nonetheless has the same effect from the point of view of proof complexity. The main result of this work is an algebraic characterisation of situations when a tree-like system can polynomially simulate its dag-like counterpart. Other results include (1) that a particular deep inference rule, cocontraction, is equivalent to dagness, from the point of view of proof complexity; and (2) a generalisation of Krajíček's result that cut can simulate the behaviour of dagness in Frege systems.

1. Introduction

Compression mechanisms in proof systems are a way to shorten proofs. They introduce natural techniques to aid a mathematical argument, for example using an already proved result again (dagness), invoking a lemma (cut) or substituting variables in a theorem (substitution). We observe, in particular, the property that compression mechanisms can, for a tautology $A \wedge B$, detect similarities between A and B and utilise them to create a proof that is often smaller than just a proof of A and a proof of B.

For example, a system needs only to prove A to conclude $A \wedge A$ in the presence of dagness, and a system with substitution can further conclude $A \wedge A\sigma$, where σ is some substitution of formulae for propositional variables. With cut it is not so obvious to see how this happens, but as an example we can simulate dagness when cut is present, due to a result of Krajíček [Kra94] which we generalise in Sect. 3.3.

Deep inference is a relatively recent proof methodology that allows inference rules to operate on arbitrary connectives appearing in a formula, rather than just the main connective like in traditional systems. [Gug07], [BT01]

One may argue that the introduction of deep inference into a proof system is just yet another example of a compression mechanism. In particular such shortening of proofs cannot be simulated in Gentzen sequent calculi unless cut is present [BG09]. We would argue against this position, on the grounds that deep inference, in the absence of dagness, cut and substitution, cannot detect similarities between conjuncts of a conjunction; any proof of $A \wedge B$ can be 'partitioned' into a proof of A and a proof of B, as shown in Thm. 37.

In this work we model formally the property that a system can detect similarities of this sort. Of course different proof systems can do this in different ways, and so we make an abstraction to complexity of proofs, which is defined for all systems [CR74]. In Sect. 4 we define the concept of an *additive norm* for a proof system, which asserts roughly that the 'size' of a minimal proof of $A \wedge B$ is equal to the sum of 'sizes' of minimal proofs of A and B, under some notion of 'size' obeying certain conditions (Sect. 4.2).

As running examples we consider Gentzen systems and deep inference systems for propositional logic; these are introduced in Sect. 3. In this section we also

Date: March 27, 2012.

observe that compression mechanisms dag and cut can be captured as inference rules, independent of any particular system, and show that they do the same job as these compression mechanisms from the point of view of proof complexity, under certain conditions on the system. A particular observation in Sect. 3 is Thm. 19, which is that a particular deep inference rule, cocontraction, is equivalent, in terms of proof complexity, to dagness.

We focus on dagness, the weakest of the compression mechanisms: its effects can be simulated by cut in Gentzen systems [Kra94], and even without cut if substitution/extension are present [Str09]. Our main result, in Sect. 4, is a characterisation of situations when tree-like systems can polynomially simulate their dag-like counterparts; we prove that a tree-like system with an additive norm can polynomially simulate its dag-like counterpart just if its dag-like counterpart also has an additive norm. A particular application of this theorem is given in Cor. 42, which states that a particular deep inference system KSg can polynomially simulate a system $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$ if and only if $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$ has an additive norm. The question of their relative complexity has previously been raised by Bruscoli, Guglielmi [BG09], Gundersen, Parigot [BGGP09], Straßburger [Str09] and Das [Das11b].

Regarding the generality of the results presented, we consider proofs systems that can be expressed in the *Calculus of Structures* (CoS) deep inference formalism. The formalism is flexible enough to embed many well-known inference systems whose basic objects are formulae, for example natural deduction, Gentzen systems, resolution and Frege systems. What cannot be embedded are systems based fundamentally on different objects, such as cutting planes and polynomial calculus, whose basic objects are algebraic inequalities. It should be noted that embeddings may not preserve all important properties of a system, particularly since there is no distinction between object and meta levels in deep inference. However what is important in this work is that proof systems can be embedded in a form that is *polynomially equivalent*. This is discussed further in Sect. 3.

Our notion of norm is related to its usual algebraic definition, in the sense that they are norms over the set TAUT of propositional tautologies, construed as a module over $\mathbb{N}$, with addition and scalar multiplication construed as conjunction.

2. Preliminaries

We briefly introduce deep inference and its usual cut-free proof systems for propositional logic. A thorough account can be found in [Brü04] and standard proof complexity results in [BG09].

Definition 1 (Formulae and Contexts)**.** The language of propositional deep inference consists of countably many atoms, denoted a, b, c, d, possibly with subscripts and superscripts, two binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction), and an involution $a \mapsto \bar{a}$, representing negation, defined only on the set of atoms, all with their usual classical interpretations. We denote by $\bot$ the empty disjunction and by $\top$ the empty conjunction.

Formulae are built freely in the usual way, and are represented by metavariables A, B, C, D. Negation of a formula is obtained by the De Morgan laws, pushing negation to the atoms. For clarity we use square brackets for disjunctions and round brackets for conjunctions, and we omit internal and external brackets when they are not necessary. $\equiv$ is a binary relation denoting syntactic equivalence of formulae, and the *size* of a formula A, denoted $|A|$, is the number of atom occurrences in A.

A *context* is a formula with one hole appearing in place of subformulae, e.g. $a \wedge [b \vee \{\ \}]$, where the hole is indicated by $\{\ \}$. We denote general contexts by metavariables $\xi\{\ \}$, $\zeta\{\ \}$. We can fill a hole with any formula, e.g. if $\xi\{\ \} \equiv a \wedge [b \vee \{\ \}]$ and A is some formula, then $\xi\{A\}$ denotes the formula $a \wedge [b \vee A]$.

Commutativity

$$A \vee B = B \vee A$$
$$A \wedge B = B \wedge A$$

Associativity

$$[A \vee B] \vee C = A \vee [B \vee C]$$
$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

Units

$$A \vee \bot = A$$
$$A \wedge \top = A$$
$$\top \vee \top = \top$$
$$\bot \wedge \bot = \bot$$

Context closure

$$\text{if } A = B \text{ then } \xi\{A\} = \xi\{B\}$$

FIGURE 1. Equality = on formulae.

We use the notation $\mathcal{C}\{\ \}$, $\mathcal{D}\{\ \}$ to denote a conjunction (resp. disjunction) with a hole, known as *conjunctive (resp. disjunctive) contexts*. We allow these to be nested and may drop braces for clarity, i.e. we write $\mathcal{CD}\{\ \}$ instead of $\mathcal{C}\{\mathcal{D}\{\ \}\}$.

Definition 2 (Inference Rules and Derivations)**.** An *inference rule* is a binary relation on formulae decidable in time polynomial in the size of its arguments. We often represent an inference rule ρ by an expression $\rho\,\dfrac{E}{F}$ where E and F are freely built using formula and context metavariables over $\{\wedge, \vee, \bar{\cdot}\}$. An instance of ρ is $\rho\,\dfrac{A}{B}$ where A, B are formulae obtained by substituting contexts and formulae for their respective metavariables.

A *deep* inference rule is one that is closed under contexts, i.e. a rule ρ is deep if for every instance $\rho\,\dfrac{A}{B}$ and every context $\xi\{\ \}$, $\rho\,\dfrac{\xi\{A\}}{\xi\{B\}}$ is also an instance of ρ.

Similarly, we say that a rule ρ is closed under conjunctions (resp. disjunctions) if for every instance $\rho\,\dfrac{A}{B}$ and every conjunctive context $\mathcal{C}\{\ \}$ (resp. disjunctive context $\mathcal{D}\{\ \}$), $\rho\,\dfrac{\mathcal{C}\{A\}}{\mathcal{C}\{B\}}$ (resp. $\rho\,\dfrac{\mathcal{D}\{A\}}{\mathcal{D}\{B\}}$)is also an instance of ρ.

We define *derivations* and *premiss*, *conclusion* functions (pr, cn resp.):

(1) Every formula is a derivation, and has premiss and conclusion as itself.
(2) If Φ is a derivation and $\rho\,\dfrac{\mathsf{cn}(\Phi)}{A}$ is an instance of some inference rule ρ, then $\rho\,\dfrac{\Phi}{A}$ is a derivation with premiss $\mathsf{pr}(\Phi)$ and conclusion A.

A *proof* is a derivation with premiss $\top$. For a proof Φ, we say that Ψ is a *subproof* of Φ, written $\Psi \leq \Phi$, if $\Psi \equiv \Phi$ or $\Phi \equiv \dfrac{\Theta}{A}$ and $\Psi \leq \Theta$.

Remark 3. An inference rule is deep just if it is closed under conjunctions and disjunctions.

Definition 4 (Equality Rule)**.** We define a binary relation = on formulae by closing the equations in Fig. 1 by reflexivity, symmetry, transitivity and by applying context closure. Any application of = can be checked in polynomial time [BG09], and so = is also an inference rule. The restriction of = with closure only under $\mathcal{CD}\{\ \}$ contexts will be denoted $='$.

Definition 5 (Systems)**.** An *inference system* (or just *system*) is a finite set of inference rules. If all inference steps appearing in a derivation (resp. proof) are instances of rules in a system $\mathcal{S}$ we say that it is an $\mathcal{S}$-derivation (resp. proof).

An $\mathcal{S}$-derivation Φ with $\mathsf{pr}(\Phi) \equiv A$, $\mathsf{cn}(\Phi) \equiv B$ is written $\begin{matrix} A \\ \Phi \| \mathcal{S} \\ B \end{matrix}$, $A \vdash^{\Phi}_{\mathcal{S}} B$ or $\mathcal{S} \frac{A}{\overline{B}}$. An $\mathcal{S}$-proof Φ with $\mathsf{cn}(\Phi) \equiv B$ is written $\begin{matrix} \Phi \| \mathcal{S} \\ B \end{matrix}$ or $\vdash^{\Phi}_{\mathcal{S}} B$.

A *deep* inference system is one which has only deep inference rules.

Remark 6. Notice that we have defined an inference rule as a binary relation, rather than one of arbitrary arity. In Gentzen systems there are rules that are not binary, for example $\wedge \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}$. When embedding various proof systems in the deep inference formalism, we translate such cases by ignoring the distinction between meta and object levels, for example the former rule is translated as $\wedge \frac{(\bigvee \Gamma \vee A) \wedge (B \vee \bigvee \Delta)}{\bigvee \Gamma \vee (A \wedge B) \vee \bigvee \Delta}$.

Definition 7. TAUT is the set of propositional tautologies, and $\mathsf{PRF}_{\mathcal{S}}$ is the set of proofs of a system $\mathcal{S}$.

Definition 8 (Complexity)**.** The *size* of a derivation Φ, denoted $|\Phi|$, is the number of atom occurrences in Φ. For $\tau \in \mathsf{TAUT}$ we define $\|\tau\|_{\mathcal{S}} = \min\{|\Phi| : \vdash^{\Phi}_{\mathcal{S}} \tau\}$.

For functions $f, g : X \to \mathbb{R}^+$ we write $f \leq_p g$ just if there is some $c \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $f(x) \leq c \cdot g(x)^n$. If $f \leq_p g$ and $g \leq_p f$ then we write $f =_p g$. Note that $=_p$ forms an equivalence relation on functions with domain X, codomain $\mathbb{R}^+$.

We say that a system $\mathcal{S}$ *polynomially simulates* a system $\mathcal{T}$ if there is a polynomial transformation $f : \mathsf{PRF}_{\mathcal{T}} \to \mathsf{PRF}_{\mathcal{S}}$ such that $\mathsf{cn}(\Phi) \equiv \mathsf{cn}(f(\Phi))$. If $\mathcal{S}$ and $\mathcal{T}$ polynomially simulate eachother, we say that they are *polynomially equivalent.*

Notation 9. We may omit the system in all notation when it is clear from context.

3. Embedding Systems and Compression Mechanisms in the Formalism

In this work we only consider inference systems as defined in the previous section; consequently, by Rmk. 6, we need to be able to 'embed' proof systems in deep inference in a way that preserves basic complexity properties. Typically this can be done for any proof system over the language defined in Dfn. 1, for example Frege systems, Gentzen and natural deduction systems.

We do not formally define 'embeddings', as it is often a case-by-case pursuit to show that the embedding for a given system preserves complexity properties, and that the notion of compression mechanisms in that system coincides with the ones we provide in Dfn. 16, from the point of view of proof complexity.

For a proof system P with tree and dag variations, there are essentially two requirements to apply the results in later sections:

(1) There is a CoS system $\mathcal{S}$ that is polynomially equivalent to tree-P.
(2) Dag-P is polynomially equivalent to dag-$\mathcal{S}$, as defined in Dfn. 16.

We provide *robustness* results in Sects. 3.1 and 3.2 that act as criteria for (2).

As running examples througout this work we consider embeddings of Gentzen and deep inference systems that satisfy the two properties above.

Definition 10. We define the deep inference systems $\mathsf{KSg} = \{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow, \mathsf{s}, =\}$, and $\mathsf{SKSg} = \mathsf{KSg} \cup \{\mathsf{i}\!\uparrow, \mathsf{w}\!\uparrow, \mathsf{c}\!\uparrow\}$ and these rules are defined in Fig. 2.

We now consider an embedding of the one-sided Gentzen system $\mathsf{GS1p}$ [TS96]. Recall that we write $\mathcal{C}\{\ \}$, $\mathcal{D}\{\ \}$ for conjunctions and disjunctions (resp.) with a hole.

Structural rules *Logical* rule

$$
\mathsf{SKSg}\left\{\begin{array}{cccc}
\mathsf{i}\!\uparrow\dfrac{\xi\{A\wedge\bar{A}\}}{\xi\{\bot\}} & \mathsf{w}\!\uparrow\dfrac{\xi\{A\}}{\xi\{\top\}} & \mathsf{c}\!\uparrow\dfrac{\xi\{A\}}{\xi\{A\wedge A\}} & \\
\textit{cointeraction} & \textit{coweakening} & \textit{cocontraction} & \\
\text{or } \textit{cut} & & & \\
\mathsf{i}\!\downarrow\dfrac{\xi\{\top\}}{\xi\{A\vee\bar{A}\}} & \mathsf{w}\!\downarrow\dfrac{\xi\{\bot\}}{\xi\{A\}} & \mathsf{c}\!\downarrow\dfrac{\xi\{A\vee A\}}{\xi\{A\}} & \mathsf{s}\dfrac{\xi\{A\wedge[B\vee C]\}}{\xi\{(A\wedge B)\vee C\}} \\
\textit{interaction} & \textit{weakening} & \textit{contraction} & \textit{switch} \\
\text{or } \textit{identity} & & &
\end{array}\right\}\mathsf{KSg}
$$

FIGURE 2. Systems KSg and SKSg.

$$
\mathsf{SGe}\left\{\begin{array}{cccc}
\mathsf{mp}\dfrac{\mathcal{C}\{[\Gamma\vee\bar{A}]\wedge[A\vee\Delta]\}}{\mathcal{C}\{\Gamma\vee\Delta\}} & & \mathsf{dag}\dfrac{\mathcal{C}\{\Gamma\}}{\mathcal{C}\{\Gamma\wedge\Gamma\}} & \\
\textit{modus ponens} & & \textit{dagness} & \\
\mathsf{id}\dfrac{\mathcal{C}\{\top\}}{\mathcal{C}\{A\vee\bar{A}\}} & \mathsf{wk}\dfrac{\mathcal{C}\{\Gamma\}}{\mathcal{C}\{\Gamma\vee A\}} & \mathsf{con}\dfrac{\mathcal{C}\{\Gamma\vee A\vee A\}}{\mathcal{C}\{\Gamma\vee A\}} & \wedge\dfrac{\mathcal{C}\{[\Gamma\vee A]\wedge[B\vee\Delta]\}}{\mathcal{C}\{\Gamma\vee(A\wedge B)\vee\Delta\}} \\
\textit{identity} & \textit{weakening} & \textit{contraction} & \wedge\textit{-intro}
\end{array}\right\}\mathsf{Ge}
$$

FIGURE 3. Embeddings of One-Sided Sequent Rules in CoS

Definition 11. We define $\mathsf{Ge}=\{\mathsf{id},\mathsf{wk},\mathsf{con},\wedge,='\}$, and $\mathsf{SGe}=\mathsf{Ge}\cup\{\mathsf{mp},\mathsf{mix},\mathsf{dag}\}$ and these rules are defined in Fig. 3.

Example 12. We give Gentzen derivation and its translation to Ge.

$$
\mathsf{con}\,\dfrac{\wedge\,\dfrac{\wedge\,\dfrac{\mathsf{id}\,\dfrac{}{A,\bar A}\qquad \mathsf{wk}\,\dfrac{\mathsf{id}\,\dfrac{}{\bar B,B}}{\bar B,B\vee C}}{A,\bar A\wedge[B\vee C],\bar B}\qquad \mathsf{id}\,\dfrac{}{A,\bar A}}{A,A,\bar A\wedge[B\vee C],\bar B\wedge\bar A}}{A,\bar A\wedge[B\vee C],\bar B\wedge\bar A}
\quad\mapsto\quad
\mathsf{con}\,\dfrac{\wedge\,\dfrac{\wedge\,\dfrac{\mathsf{wk}\,\dfrac{3\cdot\mathsf{id}\,\dfrac{='\,\dfrac{\top}{\top\wedge\top\wedge\top}}{[A\vee\bar A]\wedge[\bar B\vee B]\wedge[A\vee\bar A]}}{[A\vee\bar A]\wedge[\bar B\vee B\vee C]\wedge[A\vee\bar A]}}{[A\vee(\bar A\wedge[B\vee C])\vee\bar B]\wedge[A\vee\bar A]}}{A\vee A\vee(\bar A\wedge[B\vee C])\vee(\bar B\wedge\bar A)}}{A\vee(\bar A\wedge[B\vee C])\vee(\bar B\wedge\bar A)}
$$

Remark 13. We will always consider id, wk, con, $\wedge$, $='$ to be special cases of $\mathsf{i}\!\downarrow$, $\mathsf{w}\!\downarrow$, $\mathsf{c}\!\downarrow$, s, $=$ respectively. We write $\mathsf{Ge}\subseteq\mathsf{KSg}$, as an abuse of notation.

We state the following theorem, whose proof is a straightforward induction and can be found in App. 6.

Theorem 14. *The system* Ge *is polynomially equivalent to tree-like cut-free Gentzen systems.* $\mathsf{Ge}\cup\{\mathsf{dag}\}$ *is polynomially equivalent to dag-like cut-free Gentzen systems.* $\mathsf{Ge}\cup\{\mathsf{mp}\}$ *is polynomially equivalent to tree-like Gentzen systems with cut.*

Notation 15. We often use a notation where we only indicate parts of a formula that are modified by an inference step. E.g. rather than writing $\rho\,\dfrac{C\vee A}{C\vee B}$ we write

$C \vee \rho \dfrac{A}{B}$ and, more generally, $\xi \left\{ \rho \dfrac{A}{B} \right\}$ rather than $\rho \dfrac{\xi\{A\}}{\xi\{B\}}$. We refer to this representation as *open deduction* notation; it is called *synchronous form* in [GGP10] where its existence and uniqueness is shown for any proof.

We construe compression mechanisms dag and cut as inference rules that are independent of any particular proof system. We devote the remainder of this section showing that their effects coincide with their respective notions in Gentzen and deep inference systems. We also generalise the result of Krajíček, that dag-like systems with cut can be polynomially simulated by their corresponding tree-like systems [KP89], to a variety of systems.

In [BG09] and [BGGP09] Bruscoli and Guglielmi comment that cocontraction can be considered a generalisation of dagness. For this reason we draw attention particularly to Sect. 3.1 where we prove that cocontraction in deep inference systems is, in fact, equivalent to dagness from the point of view of proof complexity.

Definition 16 (Compression Mechanisms)**.** We define the following:

$$\mathsf{cut}\ \frac{A \vee (B \wedge \bar{B})}{A} \qquad \mathsf{dag}'\ \frac{A}{A \wedge A}$$

Remark 17. In most systems (e.g. Frege, Gentzen, Resolution etc.), due to the distinction between meta and object levels, the 'dag-rule' corresponds to something more like dag, as defined for Ge in Fig. 3: $\mathsf{dag}\ \dfrac{B \wedge A}{B \wedge A \wedge A}$. The reason we consider dag' as above is twofold: (1) for minimality, we wish to use the weakest and most general notions necessary; and (2) this notion of dagness is more useful in Sect. 4 where we prove a characterisation of dag-like speedups.

We spend the remainder of this section providing robustness results, showing that the effects of the two rules above coincide with the effects of dag and cut for a variety of systems.

3.1. Equivalence of dag' and Cocontraction in Deep Inference Systems.

We exploit the depth-change trick in [Das11a] to show the equivalence of dag' and cocontraction in deep inference systems.

Definition 18. We say that a system $\mathcal{S}$ *distributes over* $\wedge$ if, for $\star \in \{\wedge, \vee\}$, there are polynomial-size derivations $(A \star B) \wedge (A \star C) \vdash A \star (B \wedge C)$ and $(B \star A) \wedge (C \star A) \vdash (B \wedge C) \star A$.

Theorem 19. *For every deep inference system $\mathcal{S}$ that distributes over $\wedge$, $\mathcal{S} \cup \{\mathsf{c}\!\uparrow\}$ is polynomially equivalent to $\mathcal{S} \cup \{\mathsf{dag}'\}$.*

As an example, consider the following transformation of a cocontraction step:

$$A \vee \left(\mathsf{c}\!\uparrow \frac{B}{B \wedge B} \wedge C \right) \quad \to \quad \begin{array}{c} \mathsf{dag}'\ \dfrac{A \vee (B \wedge C)}{\mathsf{dist}\ \dfrac{[A \vee (B \wedge C)] \wedge [A \vee (B \wedge C)]}{A \vee \mathsf{dist}\ \dfrac{(B \wedge C) \wedge (B \wedge C)}{(B \wedge B) \wedge C}}} \end{array}$$

where dist denotes the derivation of distributivity.

Proof of Thm. 19. Let dist denote the derivation of distributivity. We proceed by structural induction. The base cases are trivial, and the inductive steps are given below:

$$A \star \xi\left\{ \mathsf{c}\!\uparrow \frac{B}{B \wedge B} \right\} \quad \rightarrow \quad A \star \begin{array}{c} \mathsf{dag}' \dfrac{A \star \xi\{B\}}{(A \star \xi\{B\}) \wedge (A \star \xi\{B\})} \\ \mathsf{dist}\ \overline{\overline{\xi\{B\} \wedge \xi\{B\}}} \\ \| \\ \xi\{B \wedge B\} \end{array}$$

$$\xi\left\{ \mathsf{c}\!\uparrow \frac{B}{B \wedge B} \right\} \star A \quad \rightarrow \quad \begin{array}{c} \mathsf{dag}' \dfrac{\xi\{B\} \star A}{(\xi\{B\} \star A) \wedge (\xi\{B\} \star A)} \\ \mathsf{dist}\ \overline{\overline{\xi\{B\} \wedge \xi\{B\}}} \\ \| \\ \xi\{B \wedge B\} \end{array} \star A$$

∎

Proposition 20. KSg *distributes over* $\wedge$.

Proof. We give derivations below.

$$\mathsf{c}\!\downarrow \dfrac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{[A \vee B] \wedge [A \vee C]}{A \vee ([A \vee B] \wedge C)}}{A \vee A}}{A} \vee (B \wedge C) \qquad = \dfrac{(A \wedge B) \wedge (A \wedge C)}{\left(\mathsf{c}\!\downarrow \dfrac{\mathsf{s}\dfrac{= \dfrac{A}{A \vee \bot} \wedge A}{A \vee \left(= \dfrac{\mathsf{w}\!\downarrow \dfrac{\bot}{\top} \wedge A}{A} \right)}}{A} \right) \wedge (B \wedge C)}$$

∎

Corollary 21. $\mathsf{KSg} \cup \{\mathsf{dag}'\}$ *is polynomially equivalent to* $\mathsf{KSg} \cup \{\mathsf{c}\!\uparrow\}$.

Proof. dag' is just an instance of $\mathsf{c}\!\uparrow$ and so $\mathsf{KSg} \cup \{\mathsf{c}\!\uparrow\}$ polynomially simulates $\mathsf{KSg} \cup \{\mathsf{dag}'\}$. The other direction follows from Thm. 19 and Prop. 20. ∎

3.2. **Equivalence of Other Compression Mechanisms and their Analogues.** We show that cut is equivalent to $\mathsf{i}\!\uparrow$ in deep inference systems and mp in Gentzen systems. We show that dag' is equivalent to dag in Gentzen sytems.

Proposition 22. *If a system* $\mathcal{S}$ *is closed under conjunctions, distributes over* $\wedge$ *and contains* $='$ *and* $\wedge$, *then* $\mathcal{S} \cup \{\mathsf{cut}\}$ *is polynomially equivalent to* $\mathcal{S} \cup \{\mathsf{mp}\}$ *and* $\mathcal{S} \cup \{\mathsf{dag}'\}$ *is polynomially equivalent to* $\mathcal{S} \cup \{\mathsf{dag}\}$.

Proof. Let dist denote the derivation of distributivity. We give the following derivations of mp and dag, the latter can be considered to be just a special case of Thm. 19 by Rmk. 13.

$$\mathsf{cut}\dfrac{\wedge \dfrac{=' \dfrac{\mathcal{C}\left\{ \wedge \dfrac{[\Gamma \vee \bar{A}] \wedge [A \vee \Delta]}{\Gamma \vee (\bar{A} \wedge A) \vee \Delta} \right\}}{=' \dfrac{C}{\bot \vee C} \wedge [[\Gamma \vee \Delta] \vee (\bar{A} \wedge A)]}}{=' \dfrac{\bot \vee (C \wedge [\Gamma \vee \Delta])}{\mathcal{C}\{\Gamma \vee \Delta\}} \vee (\bar{A} \wedge A)}}{\mathcal{C}\{\Gamma \vee \Delta\}} \qquad =' \dfrac{\mathsf{dist}\ \overline{\overline{=' \dfrac{\mathsf{dag}' \dfrac{\mathcal{C}\{A\}}{\mathcal{C}\{A\} \wedge \mathcal{C}\{A\}}}{(B \wedge A) \wedge (B \wedge A)}}}}{\dfrac{B \wedge (A \wedge A)}{\mathcal{C}\{A \wedge A\}}}$$

∎

We state the following proposition whose proof is, again, considered to be just a special case of Lemma 20 by Rmk. 13.

Proposition 23. Ge *distributes over* $\wedge$.

Proof. We give derivations of distributivity below:

$$
{\scriptstyle =' }\,\dfrac{{\scriptstyle \wedge}\,\dfrac{[A \vee B] \wedge {\scriptstyle =' }\,\dfrac{A \vee C}{C \vee A}}{A \vee (B \wedge C) \vee A}}{{\scriptstyle \mathsf{con}}\,\dfrac{A \vee A}{A} \vee (B \wedge C)}
\quad , \quad
{\scriptstyle =' }\,\dfrac{(A \wedge B) \wedge (A \wedge C)}{{\scriptstyle \mathsf{con}}\,\dfrac{{\scriptstyle =' }\,\dfrac{{\scriptstyle \wedge}\,\dfrac{{\scriptstyle =' }\,\dfrac{A}{A \vee \bot} \wedge {\scriptstyle =' }\,\dfrac{A}{\bot \vee A}}{A \vee (\bot \wedge \bot) \vee A}}{A \vee A}}{A} \wedge (B \wedge C)}
$$

□

Corollary 24. $\mathsf{Ge} \cup \{\mathsf{dag}\}$ *is polynomially equivalent to* $\mathsf{Ge} \cup \{\mathsf{dag}'\}$. $\mathsf{Ge} \cup \{\mathsf{mp}\}$ *is polynomially equivalent to* $\mathsf{Ge} \cup \{\mathsf{cut}\}$.

Proof. Immediate from Props. 22 and 23. □

We also have the following proposition.

Proposition 25. *If a system* $\mathcal{S}$ *contains* $=$ *and* s *then* $\mathcal{S} \cup \{\mathsf{cut}\}$ *is polynomially equivalent to* $\mathcal{S} \cup \{\mathsf{i}\uparrow\}$.

Proof. We transform $\mathsf{i}\uparrow$ steps to cut steps by structural induction. The base case is trivial, and we give the inductive steps below:

$$
{\scriptstyle \mathsf{cut}}\,\dfrac{{\scriptstyle \mathsf{s}}\,\dfrac{A \wedge \begin{array}{c} \xi\{B \wedge \bar{B}\} \\ \Big\Vert {\scriptstyle \{\mathsf{s},=\}} \\ \xi\{\bot\} \vee (B \wedge \bar{B}) \end{array}}{(A \wedge \xi\{\bot\}) \vee (B \wedge \bar{B})}}{A \wedge \xi\{\bot\}}
\quad , \quad
{\scriptstyle \mathsf{cut}}\,\dfrac{{\scriptstyle =}\,\dfrac{A \vee \begin{array}{c} \xi\{B \wedge \bar{B}\} \\ \Big\Vert {\scriptstyle \{\mathsf{s},=\}} \\ \xi\{\bot\} \vee (B \wedge \bar{B}) \end{array}}{(A \vee \xi\{\bot\}) \vee (B \wedge \bar{B})}}{A \vee \xi\{\bot\}}
$$

□

Corollary 26. $\mathsf{KSg} \cup \{\mathsf{i}\uparrow\}$ *is polynomially equivalent to* $\mathsf{KSg} \cup \{\mathsf{cut}\}$.

3.3. **Cut Simulates the Behaviour of Dag.** We now turn to the issue of systems that are dag-like and contain some form of cut. Krajíček first noticed that tree-like Gentzen systems can polynomially simulate dag-like Gentzen systems in the presence of cut in [Kra94]. We generalise this result below. It should be noted that a similar construction appeared in [BG09] for the case of SKSg.

Theorem 27. *If a system* $\mathcal{S}$ *satisfies the following properties:*

(1) $\mathcal{S}$ *contains* id, $\wedge$ *and* $='$,
(2) $\mathcal{S}$ *is closed under conjunctions,*
(3) $\mathcal{S}$ *distributes over* $\wedge$,

then $\mathcal{S} \cup \{\mathsf{cut}\}$ *is polynomially equivalent to* $\mathcal{S} \cup \{\mathsf{dag}', \mathsf{cut}\}$.

Proof. Let dist denote the derivation of distributivity. We derive dag' in $\mathcal{S} \cup \{\mathsf{cut}\}$.

$$
\mathsf{cut}\,\dfrac{=' \,\dfrac{\wedge\,\dfrac{=' \,\dfrac{A}{\mathsf{dist}\,\dfrac{\mathsf{id}\,\dfrac{\top}{A \vee \bar{A}} \wedge \mathsf{id}\,\dfrac{\top}{A \vee \bar{A}}}{(A \wedge A) \vee \bar{A}} \wedge [A \vee \bot]}}{(A \wedge A) \vee (\bar{A} \wedge A) \vee \bot}}{(A \wedge A) \vee (\bar{A} \wedge A)}}{A \wedge A}
$$

□

Corollary 28. $\mathsf{Ge} \cup \{\mathsf{cut}\}$ *is polynomially equivalent to* $\mathsf{Ge} \cup \{\mathsf{dag}', \mathsf{cut}\}$. $\mathsf{KSg} \cup \{\mathsf{cut}\}$ *is polynomially equivalent to* $\mathsf{KSg} \cup \{\mathsf{dag}', \mathsf{cut}\}$.

The results above justify the following notation, which we will use throughout the rest of this paper.

Notation 29. We write dag-$\mathcal{S}$ to denote $\mathcal{S} \cup \{\mathsf{dag}'\}$.

4. A Characterisation of Dag-Like Speedups

We define *norms* for a system, and relate their existence to complexity-theoretic statements for that system. In particular we characterise the situations when a tree-like system is able to polynomially simulate its dag-like counterpart.

4.1. Measurings, Measures and Norms.

Before formally defining measures and measurings, we give the general idea. Recall that $\Phi \leq \Psi$ means that Φ is a subproof of Ψ.

A *measuring* is some notion of size of proof that is equal to $|\cdot|$ up to a polynomial and also monotone with respect to $\leq$ on PRF. For example $|\cdot|$ is trivially a measuring for any system. A less trivial example is given in Dfn. 36, where we define $\#\Phi$ to be the number of atom occurrences not directly after an $=$ step. However, the function

$$
M(\Phi) = \begin{cases} 1 & \Phi \equiv \quad \mathsf{w}{\downarrow}\,\dfrac{=\,\dfrac{\mathsf{i}{\downarrow}\,\dfrac{\top}{a \vee \bar{a}}}{a \vee \bar{a} \vee \bot}}{a \vee \bar{a} \vee b} \quad (\equiv \Psi) \\ |\Phi| & \text{otherwise} \end{cases}
$$

is not a measuring since $\Psi' \equiv \mathsf{i}{\downarrow}\,\dfrac{\top}{a \vee \bar{a}} \leq \Psi$ but $M(\Psi') = 2 > 1 = M(\Psi)$.

A *measure* is a function composition of a measuring and an assignment P to tautologies of their smallest proofs, up to a polynomial. This assignment should also be closed under taking subproofs, i.e. if $M(\tau) \equiv \Phi$, $\Psi \leq \Phi$ then $M(\mathsf{cn}(\Psi)) = \Psi$. In the case above, if $P(a \vee \bar{a} \vee b) \equiv \Psi$ then $P(a \vee \bar{a}) \equiv \Psi'$.

For example, if $P(\tau)$ denotes the smallest proof of τ with an even number of contraction steps, then $\# \circ P$ is a measure returning the number of atom occurrences not directly after an $=$ step in the smallest proof with an even number of contraction steps.

Note that each measuring M induces a measure returning the minimal M-value of a proof of a tautology. We denote the function returning the minimal M-proof of a tautology by $\mathsf{MIN}^M_{\mathcal{S}}$ for a system $\mathcal{S}$.

Remark 30. As above, we sometimes refer to the 'smallest' or 'M-minimal' proof satisfying certain properties. This is not necessarily unique, but such proofs can always be chosen such that closure under subproofs is preserved.

We now give the definition formally.

Definition 31. For a system $\mathcal{S}$:

(1) A *measuring* is a function $M : \mathsf{PRF}_{\mathcal{S}} \to \mathbb{N}$ such that $M =_p |\cdot|$ and M is monotone with respect to $\leq$.
(2) A *measure* is a composition $\mu = M \circ P$ where $P : \mathsf{TAUT} \to \mathsf{PRF}_{\mathcal{S}}$ such that $|P| =_p \|\cdot\|_{\mathcal{S}}$ and $P(\mathsf{TAUT})$ is left-closed under $\leq$.
(3) $\mathsf{MIN}^M_{\mathcal{S}} : \mathsf{TAUT} \to \mathsf{PRF}_{\mathcal{S}}$ mapping a tautology to its M-minimal proof.
(4) The measure *induced* by a measuring M is $M \circ \mathsf{MIN}^M_{\mathcal{S}}$.

We proceed to define *norms* on a system, which are special types of measures, and other properties of measures.

One property, *subadditivity* or the *triangle inequality*, can be thought of as asserting that a system can prove a conjunction $A \wedge B$ by providing proofs of A and B. This is a fairly natural property and usually holds whenever a system is closed under conjunctions, as is the case for Ge and subsystems of SKSg.

On the other hand a measure that is *additive* can be thought to assert that, for its system, a proof of $A \wedge B$ is nothing but a proof of A and a proof of B, and vice-versa. This relates to an observation in the introduction: systems free of compression mechanisms have additive measures.

We also consider *homogeneity*, which is the restriction of additivity to cases where $A \equiv B$. Notice the relation to dagness, where a system can only detect similarities between conjuncts that are identical.

Recall however from the introduction that, although it is useful to think about these properties in this way, these properties fundamentally only refer to size of proofs, and so do not necessarily tell us about any semantic relationship between proofs of $A \wedge B$ and proofs of A and B. Nonetheless such a relationship is given for our example systems, Gentzen sequent calculi and deep inference, in Thm. 37, in order to prove they have additive norms.

Definition 32. We call a measure μ a *norm* if it satisfies the following properties:

(1) $\mu(\overbrace{A \wedge \cdots \wedge A}^{n}) = n \cdot \mu(A)$. (*Homogeneity*)
(2) $\mu(A \wedge B) \leq \mu(A) + \mu(B)$. (*Subadditivity, or* $\triangle$)

and we define two further properties that a measure can have:

- $\mu(A \wedge B) \geq \mu(A) + \mu(B)$. (*Superadditivity, or* $\triangledown$)
- $\mu(A \wedge B) = \mu(A) + \mu(B)$. (*Additivity, or* $\diamond$)

Recall from Notation 15 the open deduction representation of a derivation.

Example 33. Let $|\Phi|'$ denote the number of atom occurrences in a derivation Φ written in open deduction notation. Then, for any system $\mathcal{S}$, the restriction of $|\cdot|'$ to $\mathsf{PRF}_{\mathcal{S}}$, denoted $|\cdot|_{\mathcal{S}}$, is a measuring.

Let $\|\cdot\|'_{\mathcal{S}}$ be the measure induced by $|\cdot|_{\mathcal{S}}$. If $\mathcal{S}$ is closed under conjunctions then, for proofs Φ of A and Ψ of B we can construct a proof $\Phi \wedge \Psi$ of $A \wedge B$ and so $\|A \wedge B\|'_{\mathcal{S}} \leq \|A\|'_{\mathcal{S}} + \|B\|'_{\mathcal{S}}$, and so $\|\cdot\|'$ satisfies $\triangle$.

Discussion 34. Note that our definition of a norm coincides with that of a norm over a module, our module being TAUT over $\mathbb{N}$, with module addition and scalar multiplication construed as conjunction.

Our notion of an additive norm coincides with that of an L1 norm if the set of tautologous disjunctions is considered to be an orthogonal set.

Lemma 35. *For both* KSg *and* Ge*, the identity rules can be restricted to the form* $\mathsf{aid}\,\dfrac{\mathcal{C}\{\top\}}{\mathcal{C}\{a \vee \bar{a}\}}$ *with only polynomial increase in proof size.*

Proof. For KSg see [Brü04], for Ge repeatedly apply $\wedge$-intro to identity steps. □

Definition 36. Define $\#\Phi$ as the number of atom occurrences in Φ, written in open deduction notation, appearing not directly after an $=$ (or $='$) step.

For $\mathcal{S} \in \{\mathsf{Ge}, \mathsf{KSg}\}$, let $P_{\mathcal{S}}$ map tautologies to their $\#$-minimal proof in $\mathcal{S}$ such that all identity steps are instances of aid as in the lemma above. Define the measure $\mathsf{At}_{\mathcal{S}} = \# \circ P_{\mathcal{S}}$.

Theorem 37. $\mathsf{At}_{\mathcal{S}}$ *is an additive norm for* $\mathcal{S} \in \{\mathsf{Ge}, \mathsf{KSg}\}$.

Proof Outline. Let $\top^A$ denote the expression obtained by substituting $\top$ for every atom in a formula A. Consider a proof $\vdash^{\Theta}_{\mathcal{S}} A \wedge B$ and construct a proof Φ of $A \wedge \top^B$ by just mimicking, bottom-up, the steps in Θ. If an atom replaced by $\top$ was introduced by a weakening step, then that step will still be valid. If it was introduced by identity then replace that step with a weakening introducing its dual. Similarly, construct a proof Ψ of $\top^A \wedge B$.

Now notice that $|\Phi|+|\Psi| = |\Theta|$, since the atom occurrences in Φ and Ψ partition the set of atom occurrences in Θ, and so we have that $\#\left(=\dfrac{\Phi}{A}\right) + \#\left(=\dfrac{\Psi}{B}\right) = \#\Theta$. It follows that $\mathsf{At}_{\mathcal{S}}$ satisfies $\triangledown$.

By the same argument as Ex. 33, $\mathsf{At}_{\mathcal{S}}$ satisfies $\vartriangle$, and so $\mathsf{At}_{\mathcal{S}}$ is an additive norm. □

Remark 38 (Atomic Flows)**.** For the reader familiar with atomic flows ([GGS10], [Gun09]), one additive norm on KSg (and so also Ge) is given by the number of edges in a minimal flow. If Φ is a proof of a conjunction $A \wedge B$ then the $\top$-substitution in the proof of Thm. 37 partitions the flow of Φ into two disjoint components, with any identity steps in Φ spanning the conjunction replaced by weakenings, therefore preserving the number of edges.

Lemma 39. *If systems* $\mathcal{S}$, $\mathcal{T}$ *are polynomially equivalent and* $\mathcal{S} \subseteq \mathcal{T}$*, then any measure for* $\mathcal{S}$ *is also a measure for* $\mathcal{T}$.

Proof. Let $\mu = M \circ P$ be a measure for $\mathcal{S}$ and choose c, n such that $M \leq c|\cdot|^n$. Define $N : \mathsf{PRF}_{\mathcal{T}} \to \mathbb{R}^+$ as follows:

$$N(\Phi) = \begin{cases} M(\Phi) & \Phi \in \mathsf{PRF}_{\mathcal{S}} \\ c|\Phi|^n & \text{otherwise} \end{cases}$$

M and $c|\cdot|^n$ are monotone with respect to $\leq$ on their domains of definition. Suppose $\Phi \in \mathsf{PRF}_{\mathcal{S}}$, $\Psi \in \mathsf{PRF}_{\mathcal{T}} \setminus \mathsf{PRF}_{\mathcal{S}}$ and $\Phi \leq \Psi$ (notice that the other way around is impossible). Then $N(\Phi) = M(\Phi) \leq c|\Phi|^n \leq c|\Psi|^n = N(\Psi)$, so N is monotone with respect to $\leq$. Since $M =_p |\cdot|_{\mathcal{S}} = |\cdot|_{\mathcal{T}}$ restricted to $\mathsf{PRF}_{\mathcal{S}}$, N is a measuring on $\mathcal{T}$.

Finally we have $P =_p \|\cdot\|_{\mathcal{S}} =_p \|\cdot\|_{\mathcal{T}}$ so $N \circ P = M \circ P = \mu$ is a measure for $\mathcal{T}$. □

Remark 40. For the above lemma, we in fact needed something weaker than $\mathcal{S} \subseteq \mathcal{T}$; we need that every instance of an inference rule in $\mathcal{S}$ is also an instance of an inference rule in $\mathcal{T}$. This can be seen as a justification for the abuse of notation in Rmk. 13. The hypothesis is, nonetheless, sufficient to prove our main theorem.

Recall that dag-$\mathcal{S}$ is shorthand for $\mathcal{S} \cup \{\mathsf{dag}'\}$, for a system $\mathcal{S}$.

Theorem 41. *Let $\mathcal{S}$ be a system with an additive norm. The following are equivalent:*

(1) *dag-$\mathcal{S}$ has a norm.*
(2) *dag-$\mathcal{S}$ has an additive norm.*
(3) *dag-$\mathcal{S}$ is polynomially equivalent to $\mathcal{S}$.*

Proof. We prove (3) implies (2) implies (1) implies (3).

(3) $\Rightarrow$ (2). By Lemma 39 we have that the additive norm on $\mathcal{S}$ is a measure for $\mathcal{T}$. All other properties are inherited.

(2) $\Rightarrow$ (1). Trivial.

(1) $\Rightarrow$ (3). Let $\mu = M \circ P$ be a norm on dag-$\mathcal{S}$. Define $Q : \mathsf{TAUT} \to \mathsf{PRF}_{\mathcal{S}}$ inductively as follows:

$$Q(\tau) \equiv \begin{cases} P(\tau) & P(\tau) \in \mathsf{PRF}_{\mathcal{S}} \\ Q(A) \wedge Q(A) & \tau \equiv A \wedge A,\ P(\tau) \equiv {}_{\mathsf{dag}'}\dfrac{P(A)}{A \wedge A} \\ {}_{\rho}\dfrac{Q(\tau')}{\tau} & P(\tau) \equiv {}_{\rho}\dfrac{P(\tau')}{\tau},\ \rho \neq \mathsf{dag}' \end{cases}$$

Recall that $|\cdot|'$ returns the number of atom occurrences in a derivation written in open deduction notation. Note that, by Dfns. 8 and 31, $|A| > 0$ implies $\mu(A) > 0$ and so $\mu(A) \geq 1$. We prove by induction that $|Q|' \leq \mu \cdot |P|$:

Base Case. If $P(\tau) \in \mathsf{PRF}_{\mathcal{S}}$ then $|Q(\tau)|' = |P(\tau)| \leq \mu(\tau) \cdot |P(\tau)|$.

Inductive Step. If $\tau \equiv (A \wedge A)$, $P(\tau) \equiv {}_{\mathsf{dag}'}\dfrac{P(A)}{A \wedge A}$ then $Q(\tau) \equiv Q(A) \wedge Q(A)$ so:

$$\begin{aligned} |Q(\tau)|' &= |Q(A)|' + |Q(A)|' \\ &\leq \mu(A) \cdot |P(A)| + \mu(A) \cdot |P(A)| && \text{by inductive hypothesis} \\ &\leq (\mu(A) + \mu(A)) \cdot |P(A)| \\ &\leq \mu(\tau) \cdot |P(A)| && \text{by homogeneity} \\ &\leq \mu(\tau) \cdot |P(\tau)| && \text{by monotonicity of } |\cdot| \end{aligned}$$

If $P(\tau) \equiv {}_{\rho}\dfrac{P(\tau')}{\tau}$, $\rho \neq \mathsf{dag}'$ then $Q(\tau) \equiv {}_{\rho}\dfrac{Q(\tau')}{\tau}$ so:

$$\begin{aligned} |Q(\tau)|' &= |Q(\tau')|' + |\tau| \\ &\leq \mu(\tau') \cdot |P(\tau')| + |\tau| && \text{by inductive hypothesis} \\ &\leq \mu(\tau) \cdot |P(\tau')| + |\tau| && \text{by monotonicity of } \mu \\ &\leq \mu(\tau) \cdot |P(\tau')| + \mu(\tau) \cdot |\tau| && \text{since } \mu(\tau) \geq 1 \text{ or } |\tau| = 0 \\ &\leq \mu(\tau) \cdot (|P(\tau')| + |\tau|) \\ &\leq \mu(\tau) \cdot |P(\tau)| \end{aligned}$$

Finally we have $\|\cdot\|_{\mathcal{S}} \leq |Q| =_p |Q|' \leq c\mu \cdot |P| =_p \|\cdot\|_{\text{dag-}\mathcal{S}}$ and so $\mathcal{S}$ polynomially simulates dag-$\mathcal{S}$. The converse is trivial since $\mathcal{S} \subseteq$ dag-$\mathcal{S}$. ☐

Corollary 42. KSg *polynomially simulates* $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$ *if and only if there is a (additive) norm for* $\mathsf{KSg} \cup \{\mathsf{c}\uparrow\}$.

Proof. Immediate from Cor. 21, Thm. 41, Lemma 39 and Rmk. 40. ☐

Proposition 43 (Statman)**.** Ge *cannot polynomially simulate dag-*Ge.

Proof. See e.g. [CK02]. ☐

Corollary 44. Ge *does not have a norm.*

Proof. Immediate from Prop. 43, Thm. 37 and Thm. 41. □

5. Conclusions & Open Problems

We presented the Calculus of Structures formalism and showed how its flexibility can be exploited to embed a variety of proof systems. In this formalism it was shown that compression mechanisms can be captured in a system-independent way that coincides with their analogues in conventional Gentzen and deep inference systems, from the point of view of proof complexity. It is not difficult to see that this can be generalised further to other types of system.

We presented a characterisation of situations when a tree-like system with an additive norm can polynomially simulate its dag-like counterpart and presented some corollaries of this characterisation.

We conclude with a list of possible future research directions.

5.1. Normability of Gentzen with Cut.

As a corollary to Thms. 41 and 3.3, it follows that either both tree-like and dag-like Gentzen systems with cut have additive norms, or neither has. In line with our arguments in the introduction, we conjecture that neither has an additive norm.

Conjecture 45. $\mathsf{Ge} \cup \{\mathsf{dag}', \mathsf{cut}\}$ *does not have an additive norm.*

Since cut can simulate the behaviour of dag it seems intuitive that the conjecture should be true, but proving it seems to be quite difficult.

5.2. Additive Norms as a Criterion for Lack of Compression Mechanisms.

We considered the problem of defining 'proof compression' from the other direction, i.e. by understanding when a system does *not* have a compression mechanism, or is *uncompressed*. We argue that having an additive norm is equivalent to not having the ability to feasibly detect similarities between conjuncts, which is a fundamental criterion for a system to be considered to have a compresssion mechanism. We propose the following definition:

Proposed Definition 46. A system is *uncompressed* just if it has an additive norm. Otherwise it has an *effective compression mechanism.*

As in Thm. 41, it may be sufficient to just demand a norm, but we feel that the above proposed definition reflects more faithfully what is going on when compression mechanisms are present. This could be considered as a working definition, which may be modified in light of new observations of compression mechanisms not discussed in this work, but it seems that the criterion of having an additive norm is at least necessary, if not also sufficient, for a system to be regarded as uncompressed.

5.3. Robustness for Analytic Deep Inference Systems.

Robustness is the property that a class of systems satisfying certain properties are polynomially equivalent. Reckhow introduced the term in [Rec76] where he proved that all Frege systems, defined roughly to be any implicationally complete set of axioms and rules, are polynomially equivalent.

While SKSg and Gentzen systems with cut fall into this general definition of Frege system, Bruscoli and Guglielmi commented that there is no analogue for 'analytic' systems [BG09]. As an example, such a robustness theorem would be useful for generalising complexity results of systems with extension or substitution but without cut explored by Straßburger in [Str09].

Bruscoli and Guglielmi have already offered a definition of analyticity for deep inference systems ([BG07] and [BG09]), and the concept of having an additive norm may be the missing step towards a robustness theorem. We provide the following idea for a theorem in future research:

Idea 47. *Every 'regular' analytic deep inference system with an additive norm is polynomially equivalent to* KSg*.*

for some definition of 'regular'.

5.4. **Other Compression Mechanisms.** We did not consider the compression mechanisms extension and substitution. Recent work by Straßburger [Str09] has formulated these mechanisms independently from cut, which was previously unexplored, and so a relevant future direction would be to characterise these mechanisms in the same way we did for cut and dag in Sect. 3. Work in this area is already under way and, following from the previous point, we believe that it will be easier to establish theorems like 41 for these mechanisms than it will be for cut, due to the clearer way these mechanisms detect similarities between conjuncts.

5.5. **Relative Complexity of Tree-Like and Dag-Like Deep Inference.** The problem of the effect of cocontraction on proof complexity in deep inference, brought up by Bruscoli, Guglielmi [BG09], Gundersen, Parigot [BGGP09], Straßburger [Str09] and Das [Das11b], remains open. The results in Sect. 3.1, however, reduce this problem to that of whether tree-KSg polynomially simulates dag-KSg.

The concepts and results developed in Sect. 4 may provide new tools to attack this problem, but in truth we think this to be unlikely. The process of showing that a system does not have an additive norm seems to be a corollary of an already proved result on proof complexity of that system. Rather, the results in this work should be considered to shed some light on the problem and provide intuitions towards proofs or refutations of the various conjectures.

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6. Appendix: Polynomial Equivalence of Gentzen Embeddings

We show here that our embedding Ge of one-sided Gentzen systems in deep inference is polynomially equivalent to $\mathsf{GS1p}$, defined below:

$$\top \frac{}{\top} \quad \mathsf{id}\frac{}{A, \bar{A}} \quad \mathsf{wk}\frac{\Gamma}{\Gamma, A} \quad \mathsf{con}\frac{\Gamma, A, A}{\Gamma, A} \quad \wedge\frac{\Gamma, A \quad B, \Delta}{\Gamma, A \wedge B, \Delta}$$

Definition 48. Define an *origin* of an occurrence of a formula in a derivation to be a direct ancestor of that formula that has no direct ancestors.

Lemma 49. *We can polynomially transform every* Ge *proof of* $\bigwedge_i^m \bigvee_j^{n_i} A_{i,j}$ *to a set of* $\mathsf{GS1p}$ *proofs of sequents* $\vdash A_{1,1}, \dots, A_{1,n_1}$ *and* $\cdots$ *and* $\vdash A_{m,1}, \dots A_{m,n_m}$.

Proof. **Base case**

$$\top \quad \to \quad \top\frac{}{\top}$$

Inductive steps If a proof is augmented by an identity step, e.g. $\mathsf{id}\dfrac{\overset{\Vert}{\mathcal{C}\{\top\}}}{\mathcal{C}\{A \vee \bar{A}\}}$, then just replace the proof of $\top$ obtained by the inductive hypothesis by an identity step:

$$\left\{\cdots \quad \overset{\Vert}{\top} \quad \cdots\right\} \quad \to \quad \left\{\cdots \quad \mathsf{id}\frac{}{A, \bar{A}} \quad \cdots\right\}$$

If a proof is augmented by a weakening step, e.g. $\mathsf{wk}\dfrac{\overset{\Vert}{\mathcal{C}\{A\}}}{\mathcal{C}\{A \vee B\}}$, then transform the Gentzen proofs obtained by the inductive hypothesis as follows:

$$\left\{\cdots \overset{\Vert}{A_1} \cdots \overset{\Vert}{A_n} \cdots\right\} \quad \to \quad \left\{\cdots \quad \mathsf{wk}\dfrac{\wedge\dfrac{\overset{\Vert}{A_1} \quad \cdots \quad \overset{\Vert}{A_n}}{A_1 \wedge \cdots \wedge A_n}}{A_1 \wedge \cdots \wedge A_n, B} \cdots\right\}$$

where $A \equiv A_1 \wedge \cdots \wedge A_n$.

If a proof is augmented by a contraction step, e.g. $\mathsf{con}\dfrac{\overset{\|}{\mathcal{C}\{\Gamma \vee A \vee A\}}}{\mathcal{C}\{\Gamma \vee A\}}$, then transform the Gentzen proofs obtained by the inductive hypothesis as follows:

$$\left\{ \cdots \overset{\|}{\Gamma, A_1, \dots A_n, A_1, \dots A_n} \cdots \right\} \quad \to \quad \left\{ \cdots \ \mathsf{con}\dfrac{\overset{\|}{\Gamma, A_1, \dots A_n, A_1, \dots A_n}}{\Gamma, A_1, \dots, A_n} \cdots \right\}$$

where $A \equiv A_1 \vee \dots \vee A_n$.

If a proof is augmented by a $\wedge$-step, e.g. $\wedge\dfrac{\overset{\|}{\mathcal{C}\{[\Gamma \vee A] \wedge [B \vee \Delta]\}}}{\mathcal{C}\{\Gamma \vee (A \wedge B) \vee \Delta\}}$, then transform the Gentzen proofs obtained by the inductive hypothesis as follows:

$$\left\{ \cdots \overset{\|}{\Gamma, A_1, \dots, A_m}, \overset{\|}{B_1, \dots, B_n, \Delta} \cdots \right\} \quad \to \quad \left\{ \cdots \ \wedge\dfrac{\vee\dfrac{\overset{\|}{\Gamma, A_1, \dots, A_m}}{\Gamma, A_1 \vee \dots \vee A_m} \quad \vee\dfrac{\overset{\|}{B_1, \dots, B_n, \Delta}}{B_1 \vee \dots \vee B_n, \Delta}}{\Gamma, [A_1 \vee \dots \vee A_m] \wedge [B_1 \vee \dots \vee B_n], \Delta} \cdots \right\}$$

where $A \equiv A_1 \vee \dots \vee A_m$ and $B \equiv B_1 \vee \dots \vee B_n$.

The commutativity and associativity instances of $=$ are just structural manipulations on sequents, so it remains to consider the unit instances of $=$:

$$=\dfrac{\mathcal{C}\{\Gamma \vee A \vee \bot\}}{\mathcal{C}\{\Gamma \vee A\}}\,1 \qquad =\dfrac{\mathcal{C}\{\Gamma \vee (A \wedge \top)\}}{\mathcal{C}\{\Gamma \vee A\}}\,2 \qquad =\dfrac{\mathcal{C}\{\Gamma \vee \top \vee \top\}}{\mathcal{C}\{\Gamma \vee \top\}}\,3 \qquad =\dfrac{\mathcal{C}\{\Gamma \vee (\bot \wedge \bot)\}}{\mathcal{C}\{\Gamma \vee \bot\}}\,4$$
$$=\dfrac{\mathcal{C}\{A\}}{\mathcal{C}\{\Gamma \vee A \vee \bot\}}\,1' \qquad =\dfrac{\mathcal{C}\{\Gamma \vee A\}}{\mathcal{C}\{\Gamma \vee (A \wedge \top)\}}\,2' \qquad =\dfrac{\mathcal{C}\{\Gamma \vee \top\}}{\mathcal{C}\{\Gamma \vee \top \vee \top\}}\,3' \qquad =\dfrac{\mathcal{C}\{\Gamma \vee \bot\}}{\mathcal{C}\{\Gamma \vee (\bot \wedge \bot)\}}\,4'$$

We consider instances 1 and 2 first and the rest will follow.

(1) From a Ge-proof of $\mathcal{C}\{\Gamma \vee A \vee \bot\}$ construct a GS1p proof π of the sequent $\Gamma, A, \bot$, by the inductive hypothesis. If the origin of $\bot$ was an identity step $\mathsf{id}\dfrac{}{\top, \bot}$ then replace it with a $\top$-step $\top\dfrac{}{\top}$ and delete all weakening and contraction steps whose active formulae are ancestors of $\bot$ to yield a proof of $\Gamma \vee A$.

(2) From a Ge-proof of $\mathcal{C}\{\Gamma \vee (A \wedge \top)\}$ construct a GS1p-proof π of the sequent $\Gamma, A \wedge \top$, by the inductive hypothesis. Any origin of $A \wedge \top$ must be the conclusion of a $\wedge$, weakening or identity step, which we transform as follows,

$$\wedge\dfrac{\Delta, A \quad \top, \Sigma}{\Delta, A \wedge \top, \Sigma} \quad \to \quad \mathsf{wk}\dfrac{\Delta, A}{\Delta, A, \Sigma} \qquad \mathsf{wk}\dfrac{\Delta}{\Delta, A \wedge \top} \quad \to \quad \mathsf{wk}\dfrac{\Delta}{\Delta, A}$$

$$\mathsf{id}\dfrac{}{A \wedge \top, \bar{A} \vee \bot} \quad \to \quad \vee\dfrac{\mathsf{wk}\dfrac{\mathsf{id}\dfrac{}{A, \bar{A}}}{A, \bar{A}, \bot}}{A, \bar{A} \vee \bot}$$

and replace every other direct ancestor of $A \wedge \top$ with A to yield a proof of Γ, A as required.

($1'$) and ($3'$) are special cases of weakening, while ($4'$) can be considered as an instance of (1) and a weakening. (3) can just be replaced by a $\top$-step and weakenings, ($2'$) by a $\top$-step and $\wedge$-intro. (4) can be considered similarly to (1). $\square$

From the above lemma the following theorem can be established, by just applying $\vee$ and $\wedge$ rules.

Theorem 50. Ge *is polynomially equivalent to cut-free tree-like Gentzen systems.*

The next corollary is obtained as an extension of the arguments above.

Corollary 51. $\mathsf{Ge} \cup \{\mathsf{dag}\}$ *is polynomially equivalent to cut-free dag-like Gentzen systems.* $\mathsf{Ge} \cup \{\mathsf{mp}\}$ *is poylnomially equivalent to Gentzen systems with cut.*