**THE SIZE OF LINEAR DERIVATIONS IN DEEP INFERENCE**

ANUPAM DAS

Abstract. In unit-free deep inference it is known that derivations comprising of just the logical rules (switch and medial) are polynomial in size. When units are thrown in this picture changes drastically, as exhibited here. Nonetheless we show that such derivations can always be converted to ones of polynomial size, preserving the premiss and conclusion, without using structural rules.

We do not give preliminaries, please consult `http://alessio.guglielmi.name/res/cos/` for a thorough introduction to deep inference. This note forms part of ongoing work with Bruscoli, Guglielmi and Straßburger.

## 1. Large Linear Derivations in the Presence of Units

In $\{\mathsf{s}, \mathsf{m}\}$, with units, one can trivially create derivations of unbounded size by adding superfluous units, e.g.

$$a \quad \to \quad \mathsf{t} \wedge a \quad \to \quad \mathsf{t} \wedge \mathsf{t} \wedge a \quad \to \quad \cdots \quad \to \quad \mathsf{t}^n \wedge a$$

These can, of course, be simply reduced via $=$ to $a$, but one can also create derivations with unboundedly many atom occurrences that cannot be locally reduces via $=$, e.g. by the following loop

$$
{=}\,\dfrac{{=}\,\dfrac{{=}\,\dfrac{{=}\,\dfrac{\mathsf{t} \vee (a \wedge b)}{\mathsf{t} \vee \mathsf{s}\,\dfrac{a \wedge {=}\,\dfrac{b}{\dfrac{\mathsf{f}}{\mathsf{t}} \vee b}}{{=}\,\dfrac{a \wedge \mathsf{t}}{a} \vee b}}}{\mathsf{t} \vee a \vee b}}{\mathsf{t} \vee 2{\cdot}\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{{=}\,\dfrac{a}{\mathsf{t} \wedge a} \vee {=}\,\dfrac{b}{\mathsf{t} \wedge b}}{[\mathsf{t} \vee a] \wedge [\mathsf{t} \vee b]}}{\mathsf{t} \vee \mathsf{t} \vee (a \wedge b)}}}{\mathsf{t} \vee (a \wedge b)}
$$

However one can argue that this freedom to build derivations as large as we please is due to the fact that atoms lie within the scope of a disjunction containing $\mathsf{t}$ or a conjunction containing $\mathsf{f}$. Let us call an atom *trivialised* if it occurs in a derivation in this manner.

We show here that one can even build derivations with exponentially many non-trivialised atoms.

---

*Date*: September 26, 2012.

1.1. **Linear Derivability of Supermix.** We present a new rule, *supermix*, that is derivable in $\{\mathsf{s}, \mathsf{m}\}$ in the presence of units, and show that one can construct exponentially long derivations with it of only non-trivialised atoms.

**Definition 1** (Supermix)**.** We define the supermix rule below:

$$\mathsf{smix}\,\dfrac{a \wedge \bigvee_i^n b_i}{a \vee \bigwedge_i^n b_i}$$

For the special case when $n = 1$, it coincides with the usual mix rule. Supermix is sound, and trivially derivable for $\{\mathsf{w}\!\downarrow, \mathsf{w}\!\uparrow\}$.

**Proposition 2.** *There is a derivation from* $\mathsf{f}$ *to* $\mathsf{t}$ *for* $\{\mathsf{m}\}$.

*Proof.*

$$= \dfrac{\mathsf{f}}{\mathsf{m}\,\dfrac{= \dfrac{\mathsf{f}}{\mathsf{f} \wedge \mathsf{t}} \vee = \dfrac{\mathsf{f}}{\mathsf{f} \wedge \mathsf{t}}}{= \dfrac{= \dfrac{\mathsf{f} \vee \mathsf{t}}{\mathsf{t}} \wedge = \dfrac{\mathsf{f} \vee \mathsf{t}}{\mathsf{t}}}{\mathsf{t}}}}$$

$\square$

**Lemma 3.** *There is a derivation from* $\bigvee_i^n b_i$ *to* $\mathsf{t} \vee \bigwedge_i^n b_i$ *for* $\{\mathsf{s}, \mathsf{m}\}$.

*Proof.* We proceed by induction on $n$.

*Base Case*: by Prop. 2 we have $\mathsf{m}\,\dfrac{\mathsf{f}}{\mathsf{t}} \vee b$.

*Inductive Step*: Suppose there are such derivations $\Phi_r$ for $r < n$. Define:

$$\Phi_n \quad \equiv \quad \mathsf{m}\,\dfrac{= \dfrac{b_n}{\mathsf{t} \wedge b_n} \vee = \dfrac{\begin{array}{c} \bigvee_i b_i \\ \Phi_{n-1} \Big\Vert \{\mathsf{s},\mathsf{m}\} \\ \mathsf{t} \vee \bigwedge_i b_i \end{array}}{\mathsf{t} \wedge [\mathsf{t} \vee \bigwedge_i b_i]}}{\mathsf{s}\,\dfrac{[\mathsf{t} \vee b_n] \wedge [\mathsf{t} \vee \mathsf{t} \vee \bigwedge_i b_i]}{= \dfrac{\mathsf{t} \vee \mathsf{t} \vee \mathsf{t}}{\mathsf{t}} \vee (b_n \wedge \bigwedge_i b_i)}}$$

$\square$

**Theorem 4.** *Supermix is derivable for* $\{\mathsf{s}, \mathsf{m}\}$.

*Proof.* Let $\Phi_n$ be the derivations constructed in Lemma 3.

$$\mathsf{s}\,\dfrac{a \wedge \begin{array}{c} \bigvee_i^n b_i \\ \Phi_n \Big\Vert \{\mathsf{s},\mathsf{m}\} \\ \mathsf{t} \vee \bigwedge_i^n b_i \end{array}}{= \dfrac{a \wedge \mathsf{t}}{a} \vee \bigwedge_i^n b_i}$$

$\square$

**Remark 5.** all derivations in this section have used only the logical rules, and so create no new vertices in an atomic flow. Their flows are just edges in parallel.

1.2. **The Exponential-Size Linear Derivations.** Note that the premiss and conclusion of a supermix step contain no trivialisd atoms. We define the derivations inductively as follows:

$$\Lambda_1 \equiv a_1 \qquad , \qquad \Lambda_{n+1} \equiv \mathsf{smix}\,\frac{a_{n+1} \wedge \begin{array}{c}\bigwedge_{i=1}^n a_i \\ \Lambda_n \Big\| \{\mathsf{smix}\} \\ \bigvee_{i=1}^n a_i\end{array}}{a_{n+1} \vee \begin{array}{c}\bigwedge_{i=1}^n a_i \\ \Lambda_n \Big\| \{\mathsf{smix}\} \\ \bigvee_{i=1}^n a_i\end{array}}$$

**Proposition 6.** *$\Lambda_n$ has size exponential in $n$.*

Note that the above derivation is optimal, in terms of length, since each application of a rule (besides =) generates a logically distinct formula and there are only $2^n$ assignments.

## 2. A Normal Form for Trivial Derivations

Throughout we will denote the system $\{\mathsf{s}, \mathsf{m}\}$ as $\mathsf{L}$ when construed with units, and $\mathsf{L}^\star$ without. We will also assume that all atoms occurring in a formula are distinct, so that the derivations are balanced.

**Theorem 7.** *Every $\mathsf{L}^\star$-derivation has size quartic in the size of its conclusion.*

*Proof Sketch.* Let $n(A)$ denote the number of $\wedge$s occurring in a formula $A$, and let $m(A)$ denote the number of pairs of atoms in $A$ whose least common connective is $\wedge$. Clearly each medial reduces the $n$-value of a formula and each switch reduces the $m$-value of a formula, while not changing the $n$-value.

Let $M = n \star m$ denote the product measure '$n$ then $m$', then each step of an $\mathsf{L}^\star$-derivation strictly reduces $M$. But $n$ is linear in the size of a formula and $m$ is quadratic, so an $\mathsf{L}^\star$-derivation can only contain a linear $\times$ quadratic = cubic number of steps. Therefore the whole derivation has quartic size. □

**Definition 8** (Trivialised Atoms)**.** In a derivation we say that an atom is *trivialised* if at any point it occurs within the scope of a disjunction containing $\mathsf{t}$ or a conjunction containing $\mathsf{f}$.

**Proposition 9.** *There are polynomial-size derivations* $\begin{array}{c}\xi\{A\}\{\mathsf{f}\} \\ \| \{\mathsf{s}\} \\ A \vee \xi\{\mathsf{f}\}\{\mathsf{f}\}\end{array}$ , $\begin{array}{c}(\mathsf{f} \wedge a) \vee \xi\{\mathsf{f}\}\{\mathsf{f}\} \\ \| \{\mathsf{m}\} \\ \xi\{\mathsf{f}\}\{\mathsf{f} \wedge a\}\end{array}$ .

*Proof.* We proceed by induction on the depth of the holes in $\xi$. The base cases are trivial, and we give the inductive steps below.

$$\mathsf{s}\,\frac{A \wedge \left[B \vee \begin{array}{c}\xi\{\mathsf{f} \wedge a\}\{\mathsf{f}\} \\ \| \{\mathsf{s}\} \\ (\mathsf{f} \wedge a) \vee \xi\{\mathsf{f}\}\{\mathsf{f}\}\end{array}\right]}{(\mathsf{f} \wedge a) \vee (A \wedge [B \vee \xi\{\mathsf{f}\}\{\mathsf{f}\}])} \qquad , \qquad \mathsf{m}\,\frac{=\dfrac{\mathsf{f} \wedge a}{\mathsf{f} \wedge \mathsf{f} \wedge a} \vee (A \wedge [B \vee \xi\{\mathsf{f}\}\{\mathsf{f}\}])}{=\dfrac{\mathsf{f} \vee A}{A} \wedge =\dfrac{(\mathsf{f} \wedge a) \vee B \vee \xi\{\mathsf{f}\}\{\mathsf{f}\}}{B \vee \begin{array}{c}(\mathsf{f} \wedge a) \vee \xi\{\mathsf{f}\}\{\mathsf{f}\} \\ \| \{\mathsf{m}\} \\ \xi\{\mathsf{f}\}\{\mathsf{f} \wedge a\}\end{array}}}$$

□

**Lemma 10.** *Let* $\begin{array}{c}\xi\{a\} \\ \| \mathsf{L} \\ \zeta\{a\}\end{array}$ *be a derivation where $a$ is trivialised. Then there is a derivation* $\begin{array}{c}\xi\{\mathsf{t} \vee a\} \\ \| \mathsf{L} \\ \zeta\{\mathsf{f} \wedge a\}\end{array}$ *whose size is at most polynomial in the size of the former derivation.*

*Proof.* There are two cases, either

$$
\begin{array}{c}
\xi\{a\}\\ \Big\Vert\,\mathsf{L}\\ F\{\mathsf{t}\vee G\{a\}\}\\ \Big\Vert\,\mathsf{L}\\ \zeta\{a\}
\end{array}
\quad\to\quad
\begin{array}{c}
\xi\{\mathsf{t}\vee a\}\\ \Vert\,\mathsf{L}\\
F\left\{ {=}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{t}\vee G\left\{ {=}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{t}\vee {=}\,\dfrac{a}{[\mathsf{t}\vee\mathsf{f}]\wedge a}}{\mathsf{t}\vee(\mathsf{f}\wedge a)}}{\mathsf{t}\vee(\mathsf{f}\wedge a)}\right\}}{\mathsf{t}\vee G\,\{\mathsf{f}\wedge a\}}}{\mathsf{t}\vee G\,\{\mathsf{f}\wedge a\}}\right\}\\ \Vert\,\mathsf{L}\\ \zeta\{\mathsf{f}\wedge a\}
\end{array}
$$

or

$$
\begin{array}{c}
\xi\{a\}\\ \Big\Vert\,\mathsf{L}\\ F\{\mathsf{f}\wedge G\{a\}\}\\ \Big\Vert\,\mathsf{L}\\ \zeta\{a\}
\end{array}
\quad\to\quad
\begin{array}{c}
\xi\{\mathsf{t}\vee a\}\\ \Vert\,\mathsf{L}\\
F\left\{ 2\cdot\mathsf{s}\,\dfrac{{=}\,\dfrac{\mathsf{f}}{\mathsf{f}\wedge\mathsf{f}}\wedge \mathsf{s}\,\dfrac{G\left\{ {=}\,\dfrac{\mathsf{s}\,\dfrac{\mathsf{t}\vee {=}\,\dfrac{a}{[\mathsf{t}\vee\mathsf{f}]\wedge a}}{\mathsf{t}\vee(\mathsf{f}\wedge a)}}{\mathsf{t}\vee(\mathsf{f}\wedge a)}\right\}}{\mathsf{t}\vee G\{\mathsf{f}\wedge a\}}}{{=}\,\dfrac{(\mathsf{f}\wedge\mathsf{t})\vee(\mathsf{f}\wedge G\{a\})}{\mathsf{f}\wedge G\{\mathsf{f}\wedge a\}}}\right\}\\ \Vert\,\mathsf{L}\\ \zeta\{\mathsf{f}\wedge a\}
\end{array}
$$

□

**Lemma 11.** *Every* $\mathsf{L}$*-derivation where no atoms occur trivialised can be transformed into an* $\mathsf{L}^\star$*-derivation with same premiss and conclusion modulo* $=$.

*Proof.* We simply reduce every line in the derivation to a unit-free formula. Since no atoms are trivialised we do not introduce any weakening/coweakening steps. We give the four possible cases below, any other combination of linear rules with units results in some atom(s) in either the premiss or conclusion being trivialised.

$$
\mathsf{s}\,\frac{A\wedge[\mathsf{f}\vee B]}{(A\wedge B)\vee\mathsf{f}} \quad\to\quad A\wedge B \qquad\qquad \mathsf{s}\,\frac{\mathsf{t}\wedge[A\vee B]}{(\mathsf{t}\wedge A)\vee B} \quad\to\quad A\vee B
$$

$$
\mathsf{m}\,\frac{(A\wedge B)\vee(\mathsf{f}\wedge\mathsf{f})}{[A\vee\mathsf{f}]\wedge[B\vee\mathsf{f}]} \quad\to\quad A\wedge B \qquad\qquad \mathsf{m}\,\frac{(A\wedge\mathsf{t})\vee(B\wedge\mathsf{t})}{[A\vee B]\wedge[\mathsf{t}\vee\mathsf{t}]} \quad\to\quad A\vee B
$$

where $\mathsf{t}$ and $\mathsf{f}$ abbreviate a disjunction containing $\mathsf{t}$ or a conjunction containing $\mathsf{f}$ respectively. □

**Theorem 12.** *Every* $\mathsf{L}$*-derivation can be transformed into another* $\mathsf{L}$*-derivation with the same premiss and conclusion, but whose size is polynomial in the size of its premiss/conclusion.*

*Proof.* Let $\Phi$ be a $\mathsf{L}$-derivation. If there are no trivialised atoms then transform it into an $\mathsf{L}^\star$-derivation by Lemma 11 which can then must have only a quartic number of switch/medial steps by Thm. 7, and so can be simplified to a polynomial-size derivation by eliminating redundant $=$-steps.

If there is a trivialised atom in $\Phi$, say $a_1$, then transform $\Phi$ as follows:

$$\begin{array}{c}\xi\{a_1\}\\ \Phi\Big\|\mathsf{L}\\ \zeta\{a_1\}\end{array} \quad\to\quad \begin{array}{c}\xi\{\mathsf{t}\vee a_1\}\\ \Phi'\Big\|\mathsf{L}\\ \zeta\{\mathsf{f}\wedge a_1\}\end{array} \quad\to\quad \mathsf{s}\,\dfrac{\xi\left\{{=}\,\dfrac{\mathsf{t}\vee {=}\,\dfrac{a_1}{\mathsf{s}\,\dfrac{[\mathsf{t}\vee\mathsf{f}]\wedge a_1}{\mathsf{t}\vee(\mathsf{f}\wedge a_1)}}}{\mathsf{t}\vee(\mathsf{f}\wedge a_1)}\right\}}{\begin{array}{c}\xi\{\mathsf{t}\vee\mathsf{f}\}\\ \Phi_1\Big\|\mathsf{L}\\ \zeta\{\mathsf{f}\wedge\mathsf{f}\}\end{array}\vee(\mathsf{f}\wedge a_1)}$$

where $\Phi'$ is obtained from $\Phi$ by Lemma 10, and $\Phi_1$ from $\Phi'$ by substituting $\mathsf{f}$ for every instance of $a_1$; the final set of switches are obtained by Prop. 9.

Now do the same for $\Phi_1$, and repeat this process until either there are no trivialised atoms in some $\Phi_k$. (Note that it is not sufficient to just do all the trivialised atoms at once, since the act of substituting $\mathsf{f}$ for $a_i$ may result in new trivialisations.)

$$\begin{array}{c}\xi\{a_1\}\cdots\{a_k\}\\ \Phi\Big\|\mathsf{L}\\ \zeta\{a_1\}\cdots\{a_k\}\end{array} \quad\to\quad \begin{array}{c}\xi\{\mathsf{t}\vee\mathsf{f}\}\cdots\{\mathsf{t}\vee\mathsf{f}\}\\ \Phi_k\Big\|\mathsf{L}\\ \zeta\{\mathsf{f}\wedge\mathsf{f}\}\cdots\{\mathsf{f}\wedge\mathsf{f}\}\end{array} \quad \vee(\mathsf{f}\wedge a_1)\vee\cdots\vee(\mathsf{f}\wedge a_k)$$

Now by Lemma 11 we can transform $\Phi_k$ to an $\mathsf{L}^\star$-derivation $\Psi$ with same premiss and conclusion modulo $=$, which we assume to have polynomial size by Thm. 7.

$$\begin{array}{c}\xi\{\mathsf{t}\vee\mathsf{f}\}\cdots\{\mathsf{t}\vee\mathsf{f}\}\\ \Phi_k\Big\|\mathsf{L}\\ \zeta\{\mathsf{f}\wedge\mathsf{f}\}\cdots\{\mathsf{f}\wedge\mathsf{f}\}\end{array} \quad\to\quad {=}\,\dfrac{\xi\{\mathsf{t}\vee\mathsf{f}\}\cdots\{\mathsf{t}\vee\mathsf{f}\}}{\begin{array}{c}A\\ \Psi\big\|\mathsf{L}^\star\\ {=}\,\dfrac{B}{\zeta\{\mathsf{f}\wedge\mathsf{f}\}\cdots\{\mathsf{f}\wedge\mathsf{f}\}}\end{array}}$$

Finally we can apply Prop. 9 to complete the transformation:

$$\begin{array}{c}\xi\{a_1\}\cdots\{a_k\}\\ \Phi\Big\|\mathsf{L}\\ \zeta\{a_1\}\cdots\{a_k\}\end{array} \quad\to\quad \begin{array}{c} \xi\left\{{=}\,\dfrac{a_1}{\mathsf{s}\,\dfrac{[\mathsf{t}\vee\mathsf{f}]\wedge a_1}{\mathsf{t}\vee(\mathsf{f}\wedge a_1)}}\right\}\cdots\left\{{=}\,\dfrac{a_k}{\mathsf{s}\,\dfrac{[\mathsf{t}\vee\mathsf{f}]\wedge a_k}{\mathsf{t}\vee(\mathsf{f}\wedge a_k)}}\right\} \\ \mathsf{s}\ \overline{\overline{\qquad\qquad\qquad\qquad}} \\ \vdots \\ \mathsf{s}\ \overline{\overline{\qquad\qquad\qquad\qquad}} \\ {=}\,\dfrac{\xi\{\mathsf{t}\vee\mathsf{f}\}\cdots\{\mathsf{t}\vee\mathsf{f}\}}{\begin{array}{c}A\\ \Psi\big\|\mathsf{L}^\star\\ {=}\,\dfrac{B}{\zeta\{\mathsf{f}\wedge\mathsf{f}\}\cdots\{\mathsf{f}\wedge\mathsf{f}\}}\end{array}} \quad \vee(\mathsf{f}\wedge a_1)\vee\cdots\vee(\mathsf{f}\wedge a_k) \\ \mathsf{m}\ \overline{\overline{\qquad\qquad\qquad\qquad}} \\ \vdots \\ \mathsf{m}\ \overline{\overline{\qquad\qquad\qquad\qquad}} \\ \zeta\left\{{=}\,\dfrac{\dfrac{\mathsf{f}}{\mathsf{t}}\wedge a_1}{a_1}\right\}\cdots\left\{{=}\,\dfrac{\dfrac{\mathsf{f}}{\mathsf{t}}\wedge a_k}{a_k}\right\}\end{array}$$

$\square$

**Corollary 13.** *The flow of any derivation can be lifted to a derivation with same premiss/conclusion, but whose size is polynomial in the size of the flow.*