Kreisel's Conjecture and Reflexion Principles: Two Variations of the Conjecture *

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1 Preliminaries

Kreisel's Conjecture [Fri75] is the statement: if, for all $n \in \mathbb{N}$, $\mathrm{PA} \vdash_k \varphi(\overline{n})$, then $\mathrm{PA} \vdash \forall x.\varphi(x)$. Here '$\mathrm{PA} \vdash_k \varphi$' means that there is a proof in PA of φ with length at most k. There have been several partial solutions to Kreisel's Conjecture for specific system (see [Miy80] and [BP93]). In this paper we will present and prove two variations of Kreisel's Conjecture for a specific theory that extends PA.

Following [Smi13, p.101], we say that a function f is *fully-captured (as a function)* by $\varphi(x, y)$ in a theory of arithmetic T if:

(i) $T \vdash \forall x.\exists! y.\varphi(x, y)$,

and for any $m, n \in \mathbb{N}$,

(ii) If $f(m) = n$, then $T \vdash \varphi(\overline{m}, \overline{n})$;

(iii) If $f(m) \neq n$, then $T \vdash \neg\varphi(\overline{m}, \overline{n})$.

It is important to observe that if a function f is fully-captured by a Σ_1-formula $\varphi(x, y)$ in a theory T, then f is provably recursive in T (see [RS18] for a definition of this last concept). Furthermore, it is not hard to see that (iii) is implied by (i) and (ii). It is a well-known fact [Smi13, p.116] that PA can fully-capture every primitive recursive function.

We will consider Feferman's dot notation [Bar93, p.837]: considering a function-symbol $\mathrm{sub}(x, y)$ such that, for all term t, $\mathrm{PA} \vdash \mathrm{sub}(\ulcorner\varphi\urcorner, \ulcorner t\urcorner) = \ulcorner\varphi(t)\urcorner$, and a function-symbol $\mathrm{num}(x)$ that represents the numerals in PA, we define $\mathrm{s}(x, y) := \mathrm{sub}(x, \mathrm{num}(y))$ and we denote $\mathrm{s}(\ulcorner\varphi(x)\urcorner, y)$ by $\ulcorner\varphi(\dot{y})\urcorner$.

In what follows, we are going to assume that T is PA extended by the following axiom schema: if f is a primitive recursive function such that, for all $n \in \mathbb{N}$, $f(n) \neq 0$, and $\mathrm{R}(x, y)$ is a formula that fully-captures f in T, then $\mathrm{T} \vdash \forall x.\neg\mathrm{R}(x, 0)$. The results that we are going to state are also valid for T obtained from the theory $\mathrm{I}\Sigma_1$, but we decided to define for PA because the original formulation of Kreisel's Conjecture was for PA.

Given a primitive recursive function h, we will use the notation $\mathrm{PA} \vdash_{\leq h} \varphi$ to express that φ is provable in PA with a proof whose code is at most $h(\#\varphi)$. For a partial recursive function f, we use the similar notation $\mathrm{PA} \vdash_{\leq f} \varphi$ to express that $f(\#\varphi)$ is defined and φ is provable in PA with a proof whose code is at most $f(\#\varphi)$ (we will use h to denote a generic primitive recursive function and f to denote a generic partial recursive function). We will study in detail the first relation and we will establish a connection between the second relation and '$\vdash_k$'.

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We are going to prove that, given a primitive recursive function h, there is a provability predicate $\mathcal{P}_h(x)$ that expresses '$\vdash_{\leq h}$' in PA such that if, for all $n \in \mathbb{N}$, PA $\vdash_{\leq h} \alpha(\overline{n})$, then $\mathrm{T} \vdash \forall x.\mathcal{P}_h(\ulcorner \alpha(\dot{x}) \urcorner)$. From the former result, we will prove two variations of Kreisel's Conjecture.

It is important to observe that although T is stronger than PA, the axioms that are being added are Π_1-sentences of PA that arise from primitive recursive functions—we are far from adding to PA all the instances of Kreisel Conjecture; we have no *a priori* reason to suspect that the result that we are going to prove holds just because some Π_1-sentences where added to PA.

2 Main Results

The next result will play a major role in the proof of the variations of Kreisel's Conjecture. Nevertheless, it is interesting by its own right because it proves that if, for all $n \in \mathbb{N}$, PA $\vdash_{\leq h} \alpha(\overline{n})$, then one can prove the formalised version of this statement, i.e., there is a provability predicate $\mathcal{P}_h(x)$ in PA—that depends on h—such that $\mathrm{T} \vdash \forall x.\mathcal{P}_h(\ulcorner \alpha(\dot{x}) \urcorner)$.

Theorem 1. *Given a primitive recursive function h, there is a provability predicate $\mathcal{P}_h(x)$ in* PA *such that if, for all $n \in \mathbb{N}$,* PA $\vdash_{\leq h} \alpha(\overline{n})$*, then* $\mathrm{T} \vdash \forall x.\mathcal{P}_h(\ulcorner \alpha(\dot{x}) \urcorner)$.

***Proof*.** Let us consider h a fixed arbitrary primitive recursive function. Let us take the function f_h given by:

$$f_h(n) := \begin{cases} \mu m \leq h(n)[m \text{ is the code of a proof of the formula} & \\ \text{coded by } n], & \text{if there is such an } m \\ 0, & \text{otherwise} \end{cases}$$

and $f_h(i) := 0$, if i is not the code of a formula, where μ denotes the minimisation function (see [Bar93, p.833] for further details on minimisation). It is important to observe that the code of a proof is never 0, hence f_h is defined without overlap. We have that f_h is a primitive recursive function, thus f_h can be fully-captured by a formula $\mathrm{R}_h(x, y)$ in PA. Given $n, m \in \mathbb{N}$, it is clear that $m \leq h(n)$ is the smallest code of a proof of the formula whose code is n if, and only if, PA $\vdash \mathrm{R}_h(\overline{n}, \overline{m}) \wedge \overline{m} \neq 0$. Thus, it makes sense to consider the provability predicate $\mathcal{P}_h(x) := \exists y \neq 0.\, \mathrm{R}_h(x, y)$.

Let us suppose that, for all $n \in \mathbb{N}$, PA $\vdash_{\leq h} \alpha(\overline{n})$. Let us consider the function g_h defined by:

$$g_h(n) := f_h(\#\alpha(\overline{n})).$$

It is clear that g_h is primitive recursive. Furthermore, g_h is fully-captured by the formula $\mathrm{S}_h(x, y) := \mathrm{R}_h(\ulcorner \alpha(\dot{x}) \urcorner, y)$ since:

(i) If $g_h(n) = m$, then $f_h(\#\alpha(\overline{n})) = m$, and thus PA $\vdash \mathrm{R}_h(\ulcorner \alpha(\overline{n}) \urcorner, \overline{m})$, i.e., PA $\vdash \mathrm{S}_h(\overline{n}, \overline{m})$;

(ii) If $g_h(n) \neq m$, then $f_h(\#\alpha(\overline{n})) \neq m$, and thus PA $\vdash \neg\, \mathrm{R}_h(\ulcorner \alpha(\overline{n}) \urcorner, \overline{m})$, i.e., PA $\vdash \neg\, \mathrm{S}_h(\overline{n}, \overline{m})$;

(iii) As PA $\vdash \forall x.\exists! y.\, \mathrm{R}_h(x, y)$ it follows that PA $\vdash \forall x.\exists! y.\, \mathrm{R}_h(\ulcorner \alpha(\dot{x}) \urcorner, y)$, i.e., PA $\vdash \forall x.\exists! y.\, \mathrm{S}_h(x, y)$.

By hypothesis, for all $n \in \mathbb{N}$, there is $m \leq h(\#\alpha(\overline{n}))$ such that m is the code of a proof of $\alpha(\overline{n})$ in PA. Hence, for all $n \in \mathbb{N}$, $g_h(n) \neq 0$. As $\mathrm{S}_h(x, y)$ fully-captures g_h, we have by hypothesis that

$\mathrm{T} \vdash \forall x. \neg \mathrm{S}_h(x, 0)$. From $\mathrm{PA} \vdash \forall x. \exists! y. \mathrm{S}_h(x, y)$ follows that $\mathrm{PA} \vdash \forall x. \exists y. \mathrm{S}_h(x, y)$. Together with $\mathrm{T} \vdash \forall x. \neg \mathrm{S}_h(x, 0)$, it follows that $\mathrm{T} \vdash \forall x. \exists y \neq 0. \mathrm{S}_h(x, y)$, i.e., $\mathrm{T} \vdash \forall x. \exists y \neq 0. \mathrm{R}_h(\ulcorner \alpha(\dot{x}) \urcorner, y)$. So, $\mathrm{T} \vdash \forall x. \mathcal{P}_h(\ulcorner \alpha(\dot{x}) \urcorner)$.

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In what follows, we will use the notations introduced in the previous proof. The following result confirms that $\mathcal{P}_h(x)$ that expresses '$\vdash_{\leq h}$' in PA.

Theorem 2. *Given a primitive recursive function h, and a formula φ, we have that*

$$\mathrm{PA} \vdash_{\leq h} \varphi \iff \mathrm{PA} \vdash \mathcal{P}_h(\ulcorner \varphi \urcorner).$$

Proof. Suppose that $\mathrm{PA} \vdash_{\leq h} \varphi$. Then, considering $m := f_h(\#\varphi)$, we have that $m \neq 0$. Thus, $\mathrm{PA} \vdash \mathrm{R}_h(\ulcorner \varphi \urcorner, \overline{m}) \wedge \overline{m} \neq 0$, and so $\mathrm{PA} \vdash \exists y \neq 0. \mathrm{R}_h(\ulcorner \varphi \urcorner, y)$. Hence, $\mathrm{PA} \vdash \mathcal{P}_h(\ulcorner \varphi \urcorner)$.

Now suppose that $\mathrm{PA} \vdash \mathcal{P}_h(\ulcorner \varphi \urcorner)$. Consider $m := f_h(\#\varphi)$. It is clear that if $m \neq 0$, then $\mathrm{PA} \vdash_{\leq h} \varphi$. Suppose, aiming a contradiction, that $m = 0$. We have that $\mathrm{T} \vdash \exists y \neq 0. \mathrm{R}_h(\ulcorner \varphi \urcorner, y)$ and $\mathrm{T} \vdash \mathrm{R}_h(\ulcorner \varphi \urcorner, 0)$. As $\mathrm{PA} \vdash \forall x. \exists! y. \mathrm{R}_h(x, y)$ we arrive at a contradiction. So, $m \neq 0$, as wanted.

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The next two results will not play a major role, but they establish a relation between '$\vdash_k$' and '$\vdash_{\leq f}$', for f a partial recursive function, and also solve an open problem.

Theorem 3. *There is a partial recursive function $f(k, n)$ such that, for all formulas φ,*

$$\mathrm{PA} \vdash_k \varphi \implies \mathrm{PA} \vdash_{\leq f(k, \cdot)} \varphi.$$

Proof. Consider the partial recursive function f defined by:

$$f(k, n) := \mu m[m \text{ is the code of a proof of the formula coded by } n \text{ with length at most } k].$$

Suppose that $\mathrm{PA} \vdash_k \varphi$. Then there is a proof of φ with length at most k. By construction, $f(k, \#\varphi)$ is defined. Furthermore, $m := f(k, \#\varphi)$ is the code of a proof of φ and $m \leq f(k, \#\varphi)$. Thus, $\mathrm{PA} \vdash_{\leq f(k, \cdot)} \varphi$.

■

The next result is presented as an interesting side remark that follows immediately using the reasoning of the previous theorem, but it does not play a major role in the main result of this paper. It is a short solution to a problem proposed by Krajíček (problem 20 from [CK93]): Is there a recursive function $f(k, \#\varphi)$ such that $\mathrm{PA} \vdash_{k \text{ steps}} \varphi \implies \mathrm{PA} \vdash_{f(k, \#\varphi) \text{ symbols}} \varphi$? Let us consider relation $R(k, \#\varphi, m)$ given by: m is the code of a proof of φ whose length is at most k. It is clear that R is primitive recursive. Consider

$$g(k, \#\varphi) := \mu m[R(k, \#\varphi, m)].$$

By construction, we have that g is recursive. Let us also consider

$$j(n) := \begin{cases} \text{number of symbols in } n, & n \text{ is the code of a proof} \\ 0, & \text{otherwise.} \end{cases}$$

We have that j is primitive recursive. Finally, take

$$f(k, \#\varphi) := j(g(k, \#\varphi)).$$

By construction, f is a recursive function. The following result confirms that f fulfils our goal.

Theorem 4. $\mathrm{PA} \vdash_{k\ steps} \varphi \implies \mathrm{PA} \vdash_{f(k,\#\varphi)\ symbols} \varphi$.

Proof. Suppose that $\mathrm{PA} \vdash_{k \text{ steps}} \varphi$. By hypothesis, there is a proof of φ whose length is at most k. Thus, there is an m such that $R(k, \#\varphi, m)$. Therefore, $g(k, \#\varphi)$ is defined. Furthermore, by construction, $g(k, \#\varphi)$ is the code of a proof of φ whose length is at most k. Hence, $j(g(k, \#\varphi))$ is defined and is the number of symbols in the proof whose code is $g(k, \#\varphi)$. In sum, $\mathrm{PA} \vdash_{f(k,\#\varphi) \text{ symbols}} \varphi$.

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It is important to observe that f is not total. We propose another question: is there a total recursive function $f(k, \#\varphi)$ such that $\mathrm{PA} \vdash_{k \text{ steps}} \varphi \implies \mathrm{PA} \vdash_{f(k,\#\varphi) \text{ symbols}} \varphi$?

Moving way from the previous side-remark, we will continue to study the relation '$\vdash_{\leq h}$' with h a primitive recursive function. Considering the provability predicate $\mathcal{P}_h(x)$ defined previously, let us consider, for an arbitrarily fixed primitive recursive function h, the Uniform Reflection Principle schema given by [Bar93, p.845]:

$$\forall x.\mathcal{P}_h(\ulcorner \varphi(\dot{x}) \urcorner) \to \forall x.\varphi(x), \qquad \varphi(x) \text{ has only } x \text{ free} \qquad (\mathrm{RFN}^h(\mathrm{PA}))$$

Given Γ an arbitrary class of formulas (for example Σ_n, Π_n, or even Δ_n), we denote by $\mathrm{RFN}^h_\Gamma(\mathrm{PA})$ the previous schema restricted to Γ-formulas. Furthermore, we use the notations $\mathrm{S}^h_\Gamma := \mathrm{T} + \mathrm{RFN}^h_\Gamma(\mathrm{PA})$ and $\mathrm{S}_\Gamma := \mathrm{T} + \bigcup_{h \text{ primitive recursive}} \mathrm{RFN}^h_\Gamma(\mathrm{PA})$. We remember that there is a deep relation between ω-consistency and reflexion principles [Bar93, p.853]. More precisely, restrictions to Π-formulas of the Uniform Reflexion Principle for the standard provability predicate are equivalent to restrictions to Σ-formulas of the Uniform ω-consistency schema

$$\forall y.(\mathrm{Pr}(\ulcorner \exists x.\varphi(x, \dot{y}) \urcorner) \to \exists x.\neg \mathrm{Pr}(\ulcorner \neg\varphi(\dot{x}, \dot{y}) \urcorner)), \qquad (\omega\text{-CON})$$

where $\mathrm{Pr}(x)$ denotes the standard provability predicate in PA [Bar93, p.853]. It is important to observe that, by what was previously observed, we are adding ω-consistency and not ω-completeness, hence Kreisel's Conjecture is not being trivialised.

Two Variations of Kreisel's Conjecture. *We have the following two variations of Kreisel's Conjecture:*

(i) Let h be a primitive recursive function and $\alpha(x)$ be a Γ-formula such that, for all $n \in \mathbb{N}$, $\mathrm{PA} \vdash_{\leq h} \alpha(\overline{n})$. Then, $\mathrm{S}^h_\Gamma \vdash \forall x.\alpha(x)$.

(ii) Given a primitive recursive function h, if $\alpha(x)$ be a Γ-formula such that, for all $n \in \mathbb{N}$, $\mathrm{PA} \vdash_{\leq h} \alpha(\overline{n})$, then $\mathrm{S}_\Gamma \vdash \forall x.\alpha(x)$.

Proof. (i) Let h be a primitive recursive function and $\alpha(x)$ be a Γ-formula such that, for all $n \in \mathbb{N}$, $\mathrm{PA} \vdash_{\leq h} \alpha(\overline{n})$. By Theorem 1, we have that $\mathrm{T} \vdash \forall x.\mathcal{P}_h(\ulcorner \alpha(\dot{x}) \urcorner)$. Thus, by $\mathrm{RFN}^h_\Gamma(\mathrm{PA})$, it follows that $\mathrm{S}^h_\Gamma \vdash \forall x.\alpha(x)$.

(ii) Consider a primitive recursive function h, and suppose that $\alpha(x)$ is a Γ-formula such that, for all $n \in \mathbb{N}$, $\mathrm{PA} \vdash_{\leq h} \alpha(\overline{n})$. By (i) we have that $\mathrm{S}^h_\Gamma \vdash \forall x.\alpha(x)$, and so $\mathrm{S}_\Gamma \vdash \forall x.\alpha(x)$.

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It is important to observe that we have no immediate reason to believe that theory T is effectively axiomatised. This is something worth studying. We end this paper by presenting effectively axiomatisable theories that have the feature of Kreisel's Conjecture that are, unfortunately, not always sound.

Following the notations introduced in the proof of Theorem 1, let us consider for a fixed formula $\alpha(x)$ and for a fixed primitive recursive function h, K_h^α as being the theory PA $+\forall x.\neg\,\mathrm{R}_h(\ulcorner\alpha(\dot{x})\urcorner, 0) + \mathrm{RFN}_\Gamma^h(\mathrm{PA})$. It is important to observe that in general the theories K_h^α are not theories about true arithmetic, i.e., are not sound. A sufficient condition for K_h^α to be sound is the antecedent of Kreisel's Conjecture, i.e., for all $n \in \mathbb{N}$, PA $\vdash_{\leq h} \alpha(\overline{n})$. Although the theories K_h^α might not be sound, it is an interesting fact that they are effectively axiomatisable. From the previous results we have the following immediate consequence.

Corollary 1. *For every formula $\alpha(x)$ that belongs to a class Γ and every primitive recursive function h, $\mathrm{K}_h^\alpha \vdash \forall x.\alpha(x)$.*

Proof. From the proof of Theorem 1 we know that PA $+\forall x\neg\,\mathrm{R}_h(\ulcorner\alpha(\dot{x})\urcorner, 0) \vdash \forall x.\mathcal{P}_h(\ulcorner\alpha(\dot{x})\urcorner)$. Thus, by $\mathrm{RFN}_\Gamma^h(\mathrm{PA})$, we have that $\mathrm{K}_h^\alpha \vdash \forall x.\varphi(x)$.

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In this paper we studied Two Variations of Kreisel's Conjecture for the relation '$\vdash_{\leq h}$' with h a primitive recursive function. One very natural question that is interesting to study is if similar results could be obtained for the relation '$\vdash_{\leq f}$' with f a partial recursive function.

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