A Graph theoretic extension of Boolean logic

Work in Progress

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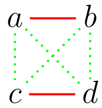
1. Introduction and motivation

In the past, there have been attempts to extend Boolean logic to a graph theoretic setting, specifically working with *relation webs* [4][5][7]. The Boolean formulas studied in this work and the aforementioned works are ones that only use the connectives $\vee, \wedge$ and no negation. The structure of relation webs allows for Boolean formulae to be considered as graphs, and motivates us to consider graphs as syntactic logical objects in there own right. Previous works have established a structural correspondence between the Boolean function of a formula and its relation web [2][4]. This result allows us to translate the semantic notions of Boolean logic into a graph theoretic setting.

For instance, take the following Boolean formula: $(a \wedge b) \vee (c \wedge d)$. The relation web of this formula is obtained by viewing the variables of the formula as the nodes of a graph, and placing an edge between two nodes if the connective at the root of the smallest subtree containing these two variables is a $\wedge$. The relation web of the formula therefore looks like this:



It is notational convention to write x —— y if there is an edge between x and y, and to write x ⋯⋯ y if there is no edge.

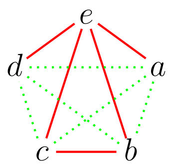
There is a natural correspondence between the minterms[1] of a linear formula and the maximal cliques of the corresponding relation web. Likewise, we have a correspondence between the maxterms of a formula and the maximal stable sets of its relation web. Coming back to the example above, we observe that the minterms of the formula are $\{\{a, b\}, \{c, d\}\}$, which coincide with the maximal cliques seen in the graph. This correspondence allows

[1] Given a Boolean formula f, a minterm (respectively maxterm) is a minimal set of variables Y such that $f(Y) = 1$ (resp. $f(\bar{Y}) = 0$).

us to define Boolean evaluation at the level of graphs, which is then naturally total and deterministic for relation webs .

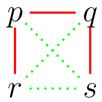
For linear formulae, we define evaluation at a graph G as evaluating an assignment Y (construed as a subset of the nodes of G) to 1 iff there is a maximal clique that is contained in Y, and evaluating to 0 iff there is a maximal stable set that is disjoint from Y. For more precise definitions, view section 2 in the appendix.

As a second, more complex example, take the formula $((a \vee (b \wedge c)) \vee d) \wedge e$. The relation web of this formula is the following:



The minterms of this formula are $\{\{a, e\}, \{d, e\}, \{c, b, e\}\}$, which coincide with the maximal cliques of the graph. The same can be said of the maxterms $\{\{a, c, d\}, \{a, b, d\}, \{e\}\}$ that coincide with the maximal stable sets of the graph.

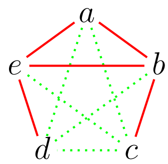
It is known that relation webs of formulae correspond only to a fairly restricted class of graphs, called *co-graphs*, equivalently P_4-*free graphs*, those graphs with no induced paths of length 4 without short-cuts [3][6]:



The aforementioned notion of evaluation on graphs allows us to study evaluation on graphs even if they do have an induced P_4. This motivates the establishment of a model of Boolean logic on arbitrary graphs, particularly those that aren't relation webs. Indeed, this gives us some interesting results. As Calk p.5 shows, the above example of a P_4 is neither deterministic: Take the assignment $Y = \{p, q\}$. Then Y evaluates to 1, since $\{p, q\}$ is a maximal clique, but also to 0, as $\{r, s\}$ is a maximal stable set that is disjoint from Y. Thus, the P_4 is not deterministic. Likewise, taking $Y = \{r, s\}$, Y evaluates neither to 0 or to 1, so the P_4 is not total.

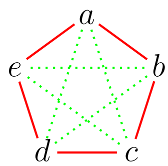
This project extends previous work concerning totality and determinism [2] of the previously established notion of evaluation in the purely graph theoretic setting and gives new direct proofs of the totality and determinism of the evaluation of P_4-free graphs. It furthermore answers the question whether these graphs are the only graphs whose evaluation fulfil both of these properties, a question which was left open by Calk [2], p.7, by showing that deterministic and total graphs are P_4-free. The proof of this is similar to a related proof found in [3], p.458.

Taking a more interesting example, we can see that there are graphs that contain a P_4 and are still deterministic. Take for instance the following:



We have the following maximal cliques: $MC_\wedge(G) = \{\{a,b,e\},\{b,c\},\{d,e\}\}$, and the following maximal stable sets: $MC_\vee(G) = \{\{a,d,c\},\{c,e\},\{b,d\}\}$. Notice that the graph does contain a P_4, namely $\{d,e,b,c\}$, yet is still deterministic. This is due to the notion of a 'settling node' [1][2], in this case is a, that resolves the previously seen issue of having a maximal clique and maximal stable set that are disjoint.

Dually, the following is an example of a graph that has a P_4, yet still computes a total evaluation:



In this example, an assignment that does not evaluate to 1 can contain at most 2 nodes. Choosing two other nodes that are not adjacent gives us a maximal stable set that is disjoint from that assignment, thus evaluating Y to 0.

The existence of such graphs motivates the search for further interesting examples and the question of how well logical phenomena of Boolean logic extend to this richer setting of arbitrary graphs, or where the natural correspondence breaks down.

In the remainder of this abstract, I go into more mathematical detail about the definitions and previous results, as well as proving the claims of determinism and totality of P_4-free graphs. The final chapter will answer the open question from Calk [2].

Bibliography

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A. Boolean logic extended to arbitrary graphs: Definitions

Definition A.1. A ***graph*** G is a couple (V, E), where V is a finite set, called the set of ***vertices*** or ***nodes***, and $E \subseteq \binom{V}{2}$ is a set of unordered pairs of distinct vertices, called the set of ***edges***. For a graph $G = (V, E)$ and a set of nodes $Y \subseteq V$ we will often just write $Y \subseteq V(G)$ or $Y \subseteq G$ instead. For $V = \{v_1, ..., v_n\}$ we call the graph $(V, \{v_1, v_2\}, ..., \{v_{k-1}, v_k\})$ a k-Path, written P_n.

Definition A.2. A subset $U \subseteq V(G)$ of a graph G is called a ***clique*** if for every $x, y \in U$, x, y are connected by an edge. Dually, $U \subseteq V(G)$ is called a ***stable set*** if for every $x, y \in U$, x, y are not connected by an edge. Given a graph G we denote the set of maximal (w.r.t inclusion) cliques (respectively stable sets) by $MC_\wedge(G)$ (resp. $MC_\vee(G)$).

Definition A.3. We define a binary relation e_G called ***evaluation*** at a graph G on $\mathcal{P}(Var) \times \{0, 1\}$ by defining:
- (i) $e_G(Y, 1)$ if $\exists S \in MC_\wedge(G)$ such that $S \subseteq Y$
- (ii) $e_G(Y, 0)$ if $\exists T \in MC_\vee(G)$ such that $T \cap Y = \emptyset$

Definition A.4. For the evaluation e_G at a graph G, we say that
- (i) e_G is ***deterministic*** if $\forall Y \subseteq V(G), \forall \alpha, \beta \in \{0, 1\}$, if $e_G(Y, \alpha)$ and $e_G(Y, \beta)$, then $\alpha = \beta$.
- (ii) e_G is ***total*** if $\forall Y \subseteq V(G) \quad \exists \alpha \in \{0, 1\}$: $e_G(Y, \alpha)$.

Definition A.5. A Graph G is called ***CIS*** if $\forall S \in MC_\wedge(G), \forall T \in MC_\vee(G), S \cap T \neq \emptyset$.

Remark. Notice that this immediately gives us $|S \cap T| = 1$ for $S \in MC(G)$, $T \in MS(G)$.

Reminder: From Calk [2] p.6 we know:

Theorem A.6. *A graph G is CIS if and only if e_G is deterministic.*

B. Determinism

Theorem B.1 (Determinism)**.** *Every non CIS graph has a P_4 configuration, i.e. every P_4-free graph is deterministic.*

Proof. Let G be a graph, and $S \in MC_\wedge(G), T \in MC_\vee(G)$ be non-empty and disjoint. Notice the following:

- (i) $\forall u \in S \quad \exists v \in T$ such that u —— v [else $T \cup \{u\}$ is maximal $\in MC_\vee(G)$]
- (ii) $\forall v \in T \quad \exists w \in S$ such that v ······ w [else $S \cup \{v\}$ is maximal $\in MC_\wedge(G)$]

Now, let $u_1 \in S$. By use of $(i), (ii)$, choose a $u_2 \in T$ such that u_1 —— u_2, then a $u_3 \in S$ such that u_2 ······ u_3 and so on. We obtain a sequence $u_1, u_2, u_3, \ldots$ such that $u_{2k+1} \in S, u_{2k} \in T$ for $k \in \mathbb{N}_0$. S and T are finite, so there must be a minimal $n \in \mathbb{N}$, such that there is an

$i \leq n$ with $u_{n+1} = u_i$, where n is either even or odd, i.e. u_n is either in T or in S.

Case 1: n is even, so $u_n \in T$, and $u_{n+1} = u_i \in S$, so $u_n \cdots u_i$. If for all odd j (i.e. $u_j \in S$) with $i \leq j < n-1$ we have $u_n \cdots u_j$, then we obtain a P_4 with the set $\{u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$. Conversely, if there is an odd j such that $i < j < n-1$ and u_n — u_j, there must be a minimal such j. But then $\{u_{j-2}, u_{j-1}, u_j, u_n\}$ is a P_4: We have $u_n \cdots u_{j-2}$ due to the minimality of j, and $u_j \cdots u_{j-1}$ due to the construction of our sequence.

Case 2: n is odd, so $u_n \in S$, and $u_{n+1} = u_i \in T$, so u_n — u_i. A similar argument works here. If for all even j (i.e. $u_j \in T$) with $i < j < n-1$ we have $u_n \cdots u_j$, then we obtain a P_4 with the set $\{u_i, u_{i+1}, u_{i+2}, u_n\}$. Conversely, if there is an even j such that $i < j < n-1$ and u_n — u_j, there must be a maximal such j. But then $\{u_j, u_{j+1}, u_{j+2}, u_n\}$ is a P_4: We have $u_n \cdots u_{j+2}$ due to the maximality of j, and $u_{j+1} \cdots u_{j+2}$ due to the construction of our sequence. □

C. Totality

Theorem C.1 (Totality). *Every P_4-free, deterministic graph is total.*

Proof. We show this by induction on the size of the graph. $|G| = 1$ is trivial. Notice the following: If any subgraph of a given graph has an induced P_4-configuration, then so does the graph itself. We now move on to the induction step.

Let $|G| = r + 1$, and assume we have proven the claim for every graph G with $|G| \leq r$. By the argument above, we know that every subgraph of G is P_4-free, and by induction hypothesis thus total. Let $Y \subseteq G$ be arbitrary, and enumerate $G \setminus Y$ by letting $G \setminus Y = \{v_1, \ldots, v_n\}$. Define the following strict subgraphs of G: $H_i = G \setminus \{v_i\}$. Notice that these are exactly those subgraphs of G that contain Y and have size r. By induction hypothesis these are total, so $e_{H_i}(Y)$ is either 0 or 1 for every $i \in \{1, \ldots, n\}$.

If $e_{H_i}(Y) = 0$ for any i, by definition there is a $T \in MC_\wedge(H_i)$ that is disjoint from Y. Either this T is a maximal stable set in G, or if not, we get $T \cup \{v_i\} \in MC_\vee(G)$. In either case, we get $e_G(Y) = 0$.

The interesting case is therefore if $e_{H_i}(Y) = 1$ for all i, and will be the case that the remaining proof focuses on. This means that for all $i \in \{1, \ldots, n\}$, there exists a $S_i \in MC_\wedge(H_i)$ such that $S_i \subseteq Y$. There are two options now. If there is an i such that $S_i \in MC_\wedge(G)$, it follows that $e_G(Y) = 1$. If there is no such i, then we immediately obtain that every $S_i \in MC_\wedge(H_i)$ extends out of Y, i.e. $S_i \cup \{v_i\} \in MC_\wedge(G)$. We denote this with S_i — v_i.

Assume now that neither $e_G(Y) = 0$ nor $e_G(Y) = 1$. We can then strengthen the above argument to get:

(i) $\forall S \in MC_\wedge(Y) \quad \exists v_i \in G \setminus Y : S$ — v_i

This relies on the fact that we exhaust $G \setminus Y$ with the $v_i's$, and that $e_G(Y) \neq 1$. Conversely, we obtain:

(ii) $\forall i \in \{1, \dots, n\} \quad \exists S \in MC_\wedge(H_i) : S \subseteq Y$ and v_i — S

(If not, then that S would still be maximal in G, and thus $e_G(Y) = 1$).

By a similar construction, we can obtain the duals to (i) and (ii). For this, enumerate Y by letting $Y = \{w_1, \dots, w_m\}$, and define subgraphs $I_i = G \setminus \{w_i\}$. Because $e_G(Y) \neq 0$, we obtain

(iii) $\forall T \in MC_\vee(G \setminus Y) \quad \exists w_i \in G : t \cdots w_i$

If this were not the case, then clearly $T \in MC_\vee(G)$ and $T \cap Y = \emptyset$ so $e_G(Y) = 0$. The converse follows using the following argument: Clearly there is no i such that $e_{I_i}(Y) = 1$, else we would also get $e_G(Y) = 1$. Because the I_i are subgraphs, they are total by induction hypothesis, so we get $e_{I_i}(Y) = 0$ for all $i \in \{1, \dots, m\}$. So for all $i \in \{1, \dots, m\}$ there is a $T_i \in MC_\vee(I_i)$ such that $T_i \cap Y = \emptyset$, and because we assume that $e_G(Y) \neq 0$, we must have $T_i \notin MC_\vee(G)$, so $w_i \cdots T_i$. We can see that this is in fact the dual to (ii) above.

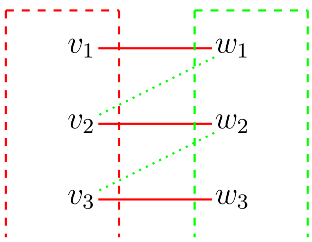
(iv) $\forall i \in \{1, \dots, m\} \quad \exists T_i \in MC_\wedge(G \setminus Y) : w_i \cdots T$

We come to the final part of the proof. Let $S \subseteq G \setminus Y$ be a maximal clique of $G \setminus Y$, and $T \subseteq Y$ be a maximal stable set of Y. Using $(i) - (iv)$ and determinism, we can obtain the following lemma:

Lemma C.2. *(a): $\forall x \in S \quad \exists y \in T$ such that x — y*
(b): $\forall y \in T \quad \exists x \in S$ such that $y \cdots x$

The proof of this is simple and symmetric for (a) and (b), we will show (a): Let $x \in S$. Then $x \in G \setminus Y$, so $x = v_i$ for some $i \leq n$. Thus by (ii) there exists a $S_i \in MC_\wedge(Y)$ such that S_i — x. By determinism $S_i \cap T \neq \emptyset$, so there exists a $y \in S_i \cap T$, and therefore x — y. The converse argument is exactly symmetrical.

We now use the lemma to generate a sequence $v_1, w_1, v_2, w_2, \dots$ such that $v_i \in S$, $w_i \in T$. Choose this sequence such that w_j is the element of T given by the lemma such that v_j — w_j, and likewise v_{j+1} is the element of S given by the lemma such that $w_j \cdots v_{j+1}$. The construction can be seen below.



Because G is finite, this sequence must loop, i.e. there is a $k \in \mathbb{N}, i \leq k$ such that $v_{k+1} = v_i$, or $w_k = w_i$. Without loss of generality, assume the former. Let $i \in \mathbb{N}$ be the maximal number such that $v_{k+1} = v_i$ and $i \leq k$. So we have v_k — w_k, $w_k \cdots v_i$, and, because i was chosen maximal, w_k — v_{i+1}. Because S, T are cliques and stable sets respectively and $v_i, v_{i+1} \in S$, $w_i, w_k \in T$, we get a P_4 with the set $\{v_i, w_i, v_{i+1}, w_k\}$.

As G is P_4-free, this is a contradiction, so our assumption that neither $e_G(Y) = 0$ nor $e_G(Y) = 1$ was wrong. It follows that G is total.

□

D. Deterministic and total graphs are P_4-free

The next theorem will be the final part in characterising P_4-free graphs, showing that a graph that is deterministic and total is P_4-free. We need to introduce an additional concept first.

Definition D.1. Let G be a graph and $Y \subseteq V(G)$. A ***selection*** with respect to Y is a set $Sel = \{T_x \mid T_x \in MS(G) \text{ and } T_x \cap Y = \{x\}\}$.
We call a selection w.r.t Y a ***covering*** if there is a $D \in MS(G)$ with $D \subseteq \cup_{x \in Y} T_x$ and $D \cap Y = \emptyset$.
We call a selection w.r.t Y a ***non-covering*** if it is not a covering.

The following Lemma is the key piece to proving the theorem:

Lemma D.2. *Let G be a graph, and $Y \subseteq V(G)$ such that there is no $S \in MC(G)$ with $Y \subseteq S$, i.e. Y is not a clique. Then every selection w.r.t Y is covering.*

Proof. We prove the contrapositive. Assume there is a non-covering selection $Sel = \{T_x \mid T_x \in MS(G) \text{ and } T_x \cap Y = \{x\}\}$ w.r.t. Y. Define the set

$$B := (V(G) \setminus \bigcup_{x \in Y} T_x) \cup Y$$

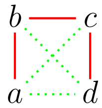
Notice that B intersects every $T \in MS(G)$: If T is a maximal stable set that intersects Y, then, as $Y \subseteq B$, T also intersects B. If T is a maximal stable set that doesn't intersect Y, then, because Sel is not a covering, we have $T \nsubseteq \cup_{x \in Y} T_x$, so T intersects B, by the definition of B.
There is no maximal stable set disjoint from B, so G doesn't evaluate B to 0. G is total, so $e_G(B, 1)$, i.e. there exists a $S \in MC(G)$ with $S \subseteq B$.

By the definition of the T_x's, we get $T_x \cap Y = \{x\}$, and by the definition of B therefore also $T_x \cap B = \{x\}$ for every $x \in Y$. We have $T_x \in MS(G)$ for every $x \in Y$, and $S \in MC(G)$, so, because G is deterministic, by the CIS property we get $|S \cap T_x| = 1$ for all $x \in Y$. Because S is contained in B, we get $S \cap T_x = \{x\}$ for all $x \in Y$. So $Y \subseteq S$. □

We now have all the results we need to prove the following lemma:

Theorem D.3 (Q3.9)**.** *A graph G is deterministic and total if and only if it is P_4-free.*

Proof. Assume G has a P_4 generated by the nodes $\{a, b, c, d\}$ like seen below.



We can extend the edges $\{a, b\}, \{c, d\}$ to maximal cliques $S_1, S_2 \in MC(G)$.

We write $S_1 = \{a, b\} \sqcup S_{ab}$, $S_2 = \{c, d\} \sqcup S_{cd}$, where $S_{ab}, S_{c,d}$ are (possibly empty) sets of nodes. Likewise, we extend the edges $\{a, c\}, \{b, d\}$ to maximal stable sets T_1, T_2, and write $T_1 = \{a, c\} \sqcup T_{ac}$, $T_2 = \{b, c\} \sqcup T_{bd}$.

Notice that by the CIS property, we get $|S_1 \cap T_1| = |S_1 \cap T_2| = |S_2 \cap T_1| = |S_2 \cap T_2| = 1$, so we have $S_1 \cap T_1 = \{a\}$, $S_1 \cap T_2 = \{b\}$, $S_2 \cap T_1 = \{c\}$, $S_2 \cap T_2 = \{d\}$. Therefore, we can easily check that the three sets $i)$ $\{a, b, c, d\}$, $ii)$ $S_{ab} \cup S_{cd}$, $iii)$ $T_{ac} \cup T_{bd}$ are all pairwise disjoint:

We show that $i), ii)$ are disjoint:
By definition, $a, b \notin S_{ab}$. Without loss of generality, assume $a \in S_{cd}$. Then $|S_2 \cap T_1| = 2$, which is a contradiction. So $a, b \notin S_{cd}$. A completely symmetric argument shows that $c, d \notin S_{ab}$, and therefore the two sets are disjoint.
The sets $i), iii)$ are disjoint by the exact same argument.

To show that $ii), iii)$ are disjoint, notice that

$$(S_{ab} \cup S_{cd}) \cap (T_{ac} \cup T_{bd}) = (S_{ab} \cap T_{ac}) \cup (S_{ab} \cap T_{bd}) \cup (S_{cd} \cap T_{ac}) \cup (S_{cd} \cap T_{bd})$$

Due to CIS, the only candidates for these intersections would be a, b, c, d, but by the previous observations, these cannot be contained in the intersection. Thus, each of the four intersections must be empty, and therefore $ii), iii)$ must be disjoint.

The set $\{a, d\}$ is not a clique, so by the previous lemma, every selection with respect to it is covering. Notice that the set $Sel = \{T_1, T_2\}$ is a selection w.r.t to $\{a, d\}$ and is therefore covering. So there is a $D \in MS(G)$ with $D \cap \{a, d\} = \emptyset$ and $D \subseteq T_1 \cup T_2$. By the previous observation, we have $(T_1 \cup T_2) \cap S_1 = \{a, c\}$, and $(T_1 \cup T_2) \cap S_2 = \{b, d\}$. D is a maximal stable set, so by CIS, $|D \cap S_1| = |D \cap S_2| = 1$, so, because $a, d \notin D$, we get $D \cap S_1 = \{c\}$, and $D \cap S_2 = \{b\}$.

So we get $b, c \in D \in MS(G)$, which is a contradiction, because there is a red edge between b and c. $\square$