On sequent calculi proofs in relevant logics

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Abstract

We present some problems in formulating cut–free sequent calculi of relevant logics, and their solutions, of which some are recent.

Some time ago, Anderson Belnap, Dunn and Gupta [1], p. 279, claimed that 'no one yet knows a decent consecution formulation of R', which 'is one of the principal relevant logics, codifying relations among $\to$, $\wedge$, $\vee$ and $\sim$'. Although some solutions for restrictions or extensions of R have appeared meanwhile, see e.g. [2], [4], [5], [6], [9], the main problem remained unsolved. Here, we shall present some reasons for that.

First, the rule of cut. This rule is indispensable for the proof of the equivalence between Hilbert–type and Gentzen-type formulations of logics. The problem is that the following form:

$$\frac{\vdash \varphi \qquad \Pi; \varphi \vdash \gamma}{\vdash \gamma}\ \text{(cut)}, \quad \text{where } \Pi \text{ is possibly empty,}$$

is needed for the proof that Modus Ponens (which is the inference rule of Hilbert–type systems), is provable in sequent calculi. Namely, to prove that from $\vdash \alpha$ and $\vdash \alpha \to \beta$ we can derive $\vdash \beta$ we proceed as follows:

$$\frac{\vdash \alpha \qquad \dfrac{\vdash \alpha \to \beta \qquad \dfrac{\alpha \vdash \alpha \qquad \beta \vdash \beta}{\alpha; \alpha \to \beta \vdash \beta}\ (\to \text{l})}{\alpha \vdash \beta}\ \text{(cut)}}{\vdash \beta}\ \text{(cut)}. \tag{1}$$

Usually, all needed forms of cut are generalized through the form:

$$\frac{\Gamma \vdash \varphi \qquad \Pi[\varphi] \vdash \gamma}{\Pi[\Gamma] \vdash \gamma}\ \text{(cut)}, \quad \text{where } \Gamma \text{ is possibly empty.} \tag{2}$$

With square brackets we indicate a specific occurrence of a sequence within a sequence, e.g. with $\Pi[\varphi]$ in (2), we emphasize a specific occurrence of the cut formula φ within a sequence Π. Moreover, $\Pi[\Gamma]$ in the lower sequent of (2) is the result of the replacement of the emphasized occurrence of φ in $\Pi[\varphi]$ by Γ.

But, (2) is exactly the form of cut which causes problems in relevant logics. Namely, in sequent systems of relevant logics, antecedents of sequents are built–up from two types of sequences, intensional, corresponding $\circ$, and extensional, corresponding to $\wedge$ (considering the terminology of Linear Logic, $\circ$ and $\wedge$ are exactly the multiplicative $\otimes$ and the additive conjunction &, respectively). We use two different punctuational marks to denote them: we use semicolons for intensional sequences and commas for extensional sequences. Those sequences

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must be allowed to be nested within one another, however nested sequences of the same kind are not allowed. So, an antecedent of our sequent can be an intensional sequence of formulae, or an intensional sequence of extensional sequences of formulae, etc., or the same thing but with intensional and extensional interchanged.

Moreover, with two different types of sequences, two different types of structural rules can also be defined, intensional and extensional. Relevant logics are Thinning–less, meaning that in their sequent systems, the structural rule of Intensional Thinning is rejected. On the other hand, they require both Extensional Thinning and Extensional Contraction, in order to enable the inference of the distribution of $\wedge$ over $\vee$, see [7], [12].

The problem which appears in relevant logics is that the cut rule (2), when Γ is empty, allows the inference of the infamous Modal Fallacy $\alpha \to (\beta \to \beta)$:

$$\dfrac{\dfrac{\vdash \beta \to \beta \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ \text{(extensional thinning)}}{\alpha \vdash \beta \to \beta}\ \text{(cut)}}{\vdash \alpha \to (\beta \to \beta)}\ (\to\ \text{r}) \tag{3}$$

Dunn [7] and Minc [12] unable the inference of Modal Fallacy by adding the truth constant t, or its functional analogue, such that in their rule of cut, $\Pi[\Gamma]$ in (2) is the result of replacing arbitrarily many occurrences of φ in $\Pi[\varphi]$ by Γ, if Γ is non–empty, and otherwise by t (instead of t Minc uses $\wedge_{i\leq m}(p_i \to p_i)$ or $\Box \wedge_{i\leq m} (p_i \to p_i)$). With Dunn's cut, one can derive:

$$\dfrac{\dfrac{\beta \vdash \beta}{\vdash \beta \to \beta}\ (\to\ \text{r}) \qquad \dfrac{\beta \to \beta \vdash \beta \to \beta}{\alpha, \beta \to \beta \vdash \beta \to \beta}\ \text{(extensional thinning)}}{\alpha, t \vdash \beta \to \beta}\ \text{(cut)}$$

but not the modal fallacy, since $\alpha, t \vdash \beta \to \beta$ cannot be the premiss of the rule $(\to$ r):

$$\dfrac{\alpha; \Gamma \vdash \beta}{\Gamma \vdash \alpha \to \beta}\ (\to\ \text{r})$$

But both strategies, Dunn's and Minc's, do not coincide with Gentzen's [8] original idea of cut, where all formulas from the premisses, except the cut formula, appear in the conclusion of cut. Moreover t is not even the part of the logic. Truly, it can be conservatively added, and after the elimination of cut, removed from the calculus, and this is exactly the procedure mostly employed, after Dunn published [7] (see e.g. [9], [5], [4], [2]). However, another solution is possible.

Simply, cut can be formulated in such a way that it can both serve to prove the equivalence between Hilbert–type and Gentzen–type systems and to disallow the inference of irrelevant formulae. This form of cut, defined in [11], has the following forms:

$$\dfrac{\Gamma \vdash \varphi \qquad \Pi[\varphi] \vdash \gamma}{\Pi[\Gamma] \vdash \gamma}\ \text{(cut−i)} \qquad \dfrac{\vdash \varphi \qquad \Pi[\varphi; \Sigma] \vdash \gamma}{\Pi[\Sigma] \vdash \gamma}\ \text{(cut−ii)} \qquad \dfrac{\vdash \varphi \qquad \varphi \vdash \gamma}{\vdash \gamma}\ \text{(cut−iii)} \tag{4}$$

where Γ is non–empty and $\Pi[\Gamma]$ in (cut-i) is the result of replacing exactly one, emphasized, occurrence of φ in $\Pi[\varphi]$ by Γ, in (cut-ii), the single occurrence of φ in $\Pi[\varphi; \Sigma]$ is replaced by an empty sequence and similarly in (cut-iii).

Although the cut rules (4) can be used in formulating sequent calculi of many positive relevant logics, the use of t remains crucial in Permutation–less logics, such as $T_{\rightarrow}$ or $E_{\rightarrow}$, where t enables the application of the rule $(B' \vdash)$ crucial for the proof of some formulae (e.g. in the proof of the restricted permutation: $(\alpha \rightarrow ((\beta \rightarrow \gamma) \rightarrow \delta)) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \delta))$ in $E_{\rightarrow}$).

As for the relevant logics with negation, the corresponding t–free formulation of cut has the following forms:

$$\frac{\vdash \Pi[\varphi] \qquad \vdash \varphi^*;\Gamma}{\vdash \Pi[\Gamma]}\ (\mathrm{cut}^*) \qquad \frac{\vdash \Pi[\varphi^*] \qquad \vdash \varphi;\Gamma}{\vdash \Pi[\Gamma]}\ (\mathrm{cut}), \quad \Gamma \text{ is non–empty}, \tag{5}$$

where φ^* can, informally be understood as the negation of φ. The star $*$ is an one–place structural connective, similar to Belnap's 'negative structuring' [3], essential in realizing the 'display' feature, but unlike Belnap's connective, the argument of our connective can be a single formula, only.

The cut rules (5) are defined in [10], where the right–handed sequent system GRW, for the Contraction–less logic RW is defined. GRW can be trivially extended to cover the full relevant logic, i.e. with this cut we can formulate a Gentzen–type system GR equivalent to the Hilbert–type formulation of R:

Axiom: $\vdash \alpha^*;\alpha$

Structural rules (Π and Σ are non–empty):

extensional contraction:
$$\frac{\vdash \Gamma[\Pi,\Pi]}{\vdash \Gamma[\Pi]}\ (\mathrm{WE})$$

extensional thinning:
$$\frac{\vdash \Gamma[\Sigma]}{\vdash \Gamma[\Pi,\Sigma]}\ (\mathrm{KE})$$

intensional contraction:
$$\frac{\vdash \Gamma[\Pi;\Pi]}{\vdash \Gamma[\Pi]}\ (\mathrm{WI})$$

cut:
$$\frac{\vdash \Gamma[\alpha] \qquad \vdash \alpha^*;\Pi}{\vdash \Gamma[\Pi]}\ (\mathrm{cut}^*) \qquad \frac{\vdash \Gamma[\alpha^*] \qquad \vdash \alpha;\Pi}{\vdash \Gamma[\Pi]}\ (\mathrm{cut})$$

Operational rules:

$$\frac{\vdash \Gamma_1;\alpha \qquad \vdash \Gamma_2[\beta^*]}{\vdash \Gamma_2[\Gamma_1;(\alpha \rightarrow \beta)^*]}\ (\rightarrow_1^*) \qquad \frac{\vdash \Gamma[\alpha^*;\beta]}{\vdash \Gamma[\alpha \rightarrow \beta]}\ (\rightarrow)$$

$$\frac{\vdash \Gamma_1[\alpha] \qquad \vdash \Gamma_2;\beta^*}{\vdash \Gamma_1[\Gamma_2;(\alpha \rightarrow \beta)^*]}\ (\rightarrow_2^*)$$

$$\frac{\vdash \Gamma[\alpha]}{\vdash \Gamma[(\sim \alpha)^*]}\ (\sim^*) \qquad \frac{\vdash \Gamma[\alpha^*]}{\vdash \Gamma[\sim \alpha]}\ (\sim)$$

$$\frac{\vdash \Gamma[\alpha^*]}{\vdash \Gamma[(\alpha \wedge \beta)^*]} \quad \frac{\vdash \Gamma[\beta^*]}{\vdash \Gamma[(\alpha \wedge \beta)^*]}\ (\wedge^*) \qquad \frac{\vdash \Gamma[\alpha] \qquad \vdash \Gamma[\beta]}{\vdash \Gamma[\alpha \wedge \beta]}\ (\wedge)$$

$$\frac{\vdash \Gamma[\alpha^*] \qquad \vdash \Gamma[\beta^*]}{\vdash \Gamma[(\alpha \vee \beta)^*]}\ (\vee^*) \qquad \frac{\vdash \Gamma[\alpha]}{\vdash \Gamma[\alpha \vee \beta]} \quad \frac{\vdash \Gamma[\beta]}{\vdash \Gamma[\alpha \vee \beta]}\ (\vee)$$

However, the appropriate cut–elimination procedure for GR is still unknown and that is what we are working on.

References

[1] Anderson, A., N. Jr. Belnap and J. M. Dunn, *Entailment: the logic of relevance and necessity,* vol. 2, Princeton University Press, Princeton, New Jersey, 1992.

[2] Belnap Jr., N.D., A. Gupta, and J. M. Dunn, *A consecutive calculus for positive relevant implication with necessity,* Journal of Philosophical Logic 9:343–362, 1980.

[3] Belnap Jr., N.D., *Display logic,* Journal of Philosophical Logic, 11:375–417, 1982.

[4] Bimbó, K. and J. M. Dunn, *New consecution calculi for* $R^t_{\rightarrow}$, Notre Dame Journal of Formal Logic, 53:491–501, 2012.

[5] Brady, R.T., *The Gentzenization and decidability of RW,* Journal of Philosophical Logic, 19:35–73, 1990.

[6] Brady, R.T., *Gentzenizations of relevant logics with distribution,* The Journal of Symbolic Logic, 61:402–420, 1996.

[7] Dunn, J.M., *A 'Gentzen system' for positive relevant implication,* The Journal of Symbolic Logic 38:356-357, 1973.

[8] Gentzen, G., *Investigations into logical deduction,* The Collected Papers of Gerhard Gentzen, North-Holland, Edited by M. E. Szabo, 68–131, 1969.

[9] Giambrone, S., TW_+ *and* RW_+ *are decidable,* Journal of Philosophical Logic, 14:235–254, 1985.

[10] Ilić, M., Boričić, B., A cut-free sequent calculus for relevant logic RW, *Logic Journal of IGPL,* 22(4): 673–695, 2014.

[11] Ilić, M., *An alternative Gentzenization of* RW°_+, Mathematical Logic Quarterly, 62(6):465–480, 2016.

[12] Minc, G., *Cut elimination theorem for relevant logics,* Journal of Soviet Mathematics, 6: 422–428, 1976.