Introduction to Proof Theory

Lectures 3 & 4 - The sequent calculus and analyticity

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These slides are available at `http://www.anupamdas.com/wp/lss18/`.

Outline

Proofs, provability and systems

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But there are many other ones to consider!

- **Natural deduction.** (Normalisation as computation)
- **Sequent calculus.** (General metalogical framework)
- **Resolution.** (Automated theorem proving)
- **Analytic tableaux.** (Semantics)
- Systems with **Extension** or **Substitution**. (Proof complexity)
- **Algebraic** and **Geometric** systems. (Structural proof theory)
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Question: Why is one system better than another? We will consider just one desideratum here...

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Proposition

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Desideratum: Proofs in $\mathcal{F}$ are rather short (believe it or not): can we trade off proof size for less nondeterminism in proof search?

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NB: If you want more of these, come speak to me!

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Gentzen's insight was to dualise this notion:

$\Gamma \vdash \Delta$: "all the formulae of Γ imply some of the formulae of Δ"

Sequents

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Definition (Sequents)

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- $\Gamma \vdash$ means "Γ is inconsistent".
- $\vdash \Delta$ means "some formula in Δ is true".

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Gentzen proposed a system called LK, which notably enjoys the following completeness property:

Theorem (Subformula property, Gentzen '34)

If A is valid, it has an LK *proof containing only subformulae of A (up to substitution of free variables for terms).*

Outline

Initial and structural rules

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- (**Contraction**, c) corresponds to the tautologies:

  $$A \to (A \wedge A) \qquad \text{and} \qquad (A \vee A) \to A$$

  Alternatively: if I already have A as a hypothesis, I do not need two copies of it!

Propositional connectives

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We have two symmetric rules for implication:

$$\to\text{-}l\,\frac{\Gamma\vdash\Delta,A\quad B,\Gamma\vdash\Delta}{\Gamma,A\to B\vdash\Delta}\qquad\to\text{-}r\,\frac{\Gamma,A\vdash B,\Delta}{\Gamma\vdash A\to B,\Delta}$$

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- The *left* rule allows us to 'compose' derivations.
- The *right* rule is just a generalisation of the deduction theorem.

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Importantly, LK has the following rules for the quantifiers:

$$\forall\text{-}l\,\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\,\frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}\; y \notin \mathsf{FV}(\Gamma, \Delta, A)$$

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$$\exists\text{-}r\,\frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\,\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}\, y \notin \mathsf{FV}(\Gamma, \Delta, A)$$

- In $\exists$-l and $\forall$-r the variable x may not occur free in Γ, Δ. (Remember Lecture 3!)
- Notice the duality between $\exists$ and $\forall$ here.

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Notice that A here, the cut-formula, had to be guessed.

This is very similar to the situation for *modus ponens* and, indeed, allows us to easily simulate $\mathcal{F}$ proofs, or prove completeness at all…

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NB: (1) may also be proved by simulation, but a little harder!

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This is one of the hardest termination arguments in graduate mathematics and computer science!

The *Hauptsatz*

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Let us at least see some of the key ideas…

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\qquad
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}{
\Gamma, \Gamma' \vdash \Delta, \Delta'
}\ \mathit{cut}
$$

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  \dfrac{\overset{1}{\vdots}\\ \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \to B}\;{\to}\text{-}r
  \qquad
  \dfrac{\overset{2}{\vdots}\\ \Gamma' \vdash \Delta', A \quad \overset{3}{\vdots}\\ \Gamma', B \vdash \Delta'}{\Gamma', A \to B \vdash \Delta'}\;{\to}\text{-}l
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;\mathit{cut}
$$

$$
\rightsquigarrow \quad
\dfrac{
  \dfrac{
    \dfrac{\overset{2}{\vdots}\ \Gamma' \vdash \Delta', A \quad \overset{1}{\vdots}\ \Gamma, A \vdash \Delta, B}{\Gamma, \Gamma' \vdash \Delta, \Delta', B}\;\mathit{cut}
    \qquad
    \overset{3}{\vdots}\ \Gamma', B \vdash \Delta'
  }{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\;\mathit{cut}
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;c
$$

Key quantifier case

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$$
\dfrac{\exists\text{-}r\ \dfrac{\overset{1}{\bigtriangledown}\\ \Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x.A(x)} \qquad \exists\text{-}l\ \dfrac{\overset{2(x)}{\bigtriangledown}\\ \Gamma', A(x) \vdash \Delta'}{\Gamma', \exists x.A(x) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathit{cut}
$$

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$$\dfrac{\dfrac{\overset{1}{\Gamma \vdash \Delta, A(t)}}{\Gamma \vdash \Delta, \exists x.A(x)}\,\exists\text{-}r \qquad \dfrac{\overset{2(x)}{\Gamma', A(x) \vdash \Delta'}}{\Gamma', \exists x.A(x) \vdash \Delta'}\,\exists\text{-}l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{1}{\Gamma \vdash \Delta, A(t)} \qquad \overset{2(t)}{\Gamma', A(t) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathit{cut}$$

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\quad\rightsquigarrow\quad
\dfrac{\overset{1}{\vdots}\\ \Gamma \vdash \Delta, A(t) \qquad \overset{2(t)}{\vdots}\\ \Gamma', A(t) \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;\textit{cut}
$$

- Implicitly assume free variable conditions are respected.
- Write $2(t)$ for the derivation arising from $2(x)$ by replacing every free occurrence of x by t.

Termination

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Proving that cut-elimination terminates is incredibly complex!

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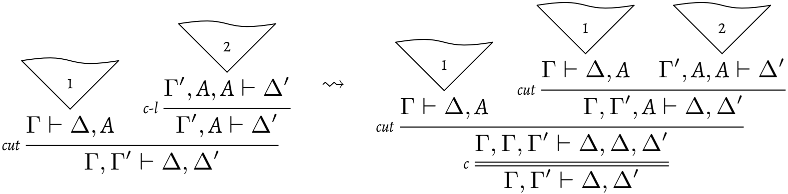
One particularly nasty issue is:



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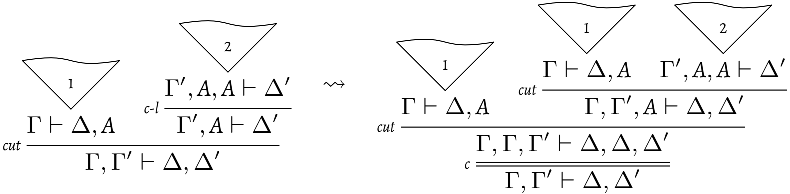

Here the entire subproof 1 is duplicated!

The need for ingenuity

Even worse, just blindly applying permutation rules will not work:

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$$
\dfrac{\dfrac{\overset{\triangledown\, 1}{\Gamma \vdash \Delta, A, A}}{\Gamma \vdash \Delta, A}\,\textit{c-r} \qquad \dfrac{\overset{\triangledown\, 2}{\Gamma', A, A \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\,\textit{c-l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\textit{cut}
\quad \rightsquigarrow \quad
\begin{array}{c}
\dfrac{\dfrac{\overset{\triangledown\, 1}{\Gamma \vdash \Delta, A, A} \qquad \overset{\triangledown\, 2}{\Gamma', A, A \vdash \Delta'}}{\Gamma, \Gamma', A \vdash \Delta, \Delta', A}\,\textit{cut}}{} \\
\vdots
\end{array}
$$

The sequent in red is trivial.

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Even worse, just blindly applying permutation rules will not work:

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The sequent in red is trivial.

This is known as **Lafont's counterexample**, cf. *non-confluence* of cut-elimination.

Outline

The contraction-tree

Definition

The **contraction-tree** of a formula occurrence A in a proof π is the tree induced by the *c-l* and *c-r* steps in π on that formula occurrence.

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A, A \vdash \Delta, B, B, B, B}{\Gamma, A, A \vdash \Delta, B, B, B}\,c\text{-}r}{\Gamma, A, A \vdash \Delta, B, B}\,c\text{-}r}{\Gamma, A \vdash \Delta, B, B}\,c\text{-}l}{\Gamma, A \vdash \Delta, B}\,c\text{-}r
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The contraction-trees of A and B are marked in blue and red respectively.

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The contraction-trees of A and B are marked in blue and red respectively.

NB: In the contraction-tree, we only count occurrences of the same formula, not its subformulae.

A crucial lemma

Lemma

For any proof

$$\textit{cut}\ \dfrac{\overset{\pi}{\Gamma \vdash \Delta, A} \qquad \overset{\pi'}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'},$$

where the subproofs π and π' are cut-free, there is a proof of the end-sequent $\Gamma, \Gamma' \vdash \Delta, \Delta'$ with only cut-formulas of size $< |A|$.

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Proof idea.

We show the strengthening that this is possible while furthermore preserving the size of contraction-trees from the end-sequent, by induction on:

$$(\text{size of } \pi') \times (\text{size of contraction-tree of } A \text{ in } \pi').$$

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The main interesting case is when π' finishes with a *c-l* step:

$$\dfrac{\Gamma \vdash \Delta, A \qquad \dfrac{\Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\,\textit{c-l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\textit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \vdash \Delta, A \qquad \dfrac{\Gamma \vdash \Delta, A \qquad \Gamma', A, A \vdash \Delta'}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}\,\textit{cut}}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}\,\textit{cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,c$$

□

An induction measure

An induction measure

Definition (Multiset measure)

For two multisets M, N of natural numbers, we say $M \prec N$ if I may transform N into M by the continually applying the following operation:

delete a number $n \in N$ *and insert any number of numbers* $< n$

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Example

$\{1, 2, 3, 4, 4, 5\} \prec \{2, 5, 5\}$ and $\{1, 1, 2, 2, 2, 2\} \prec \{1, 2, 3\}$.

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NB: This can also be seen as a lexicographic order.

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NB: This can also be seen as a lexicographic order.

Proposition

$\prec$ *is a well-order on the class of multisets of natural numbers.*

Putting it all together

Definition (Degrees)

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Proof sketch of cut-elimination.

By induction on the degree of a proof, under $\prec$:

- Identify a topmost cut in a proof, and apply the crucial lemma.
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Exercise: Try work out explicitly the induction measure for yourself!

Outline

A cut-free proof of contraposition:

$$
{\scriptstyle \to\text{-}r}\,\dfrac{{\scriptstyle \to\text{-}r}\,\dfrac{{\scriptstyle \to\text{-}l}\,\dfrac{{\scriptstyle \to\text{-}r}\,\dfrac{{\scriptstyle w}\,\dfrac{\textsf{id}\,\dfrac{}{B \vdash B}}{A, B \vdash B}}{A \vdash B, \neg B} \qquad {\scriptstyle \to\text{-}l}\,\dfrac{{\scriptstyle w}\,\dfrac{\textsf{id}\,\dfrac{}{A \vdash A}}{A \vdash A, B} \qquad {\scriptstyle w}\,\dfrac{{\scriptstyle \bot\text{-}l}\,\dfrac{}{\bot \vdash}}{\bot \vdash B}}{A, \neg A \vdash B}}{\neg B \to \neg A, A \vdash B}}{\neg B \to \neg A \vdash A \to B}}{\vdash (\neg B \to \neg A) \to A \to B}
$$

A cut-free proof of Pierce's law:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\textsf{id}}{A \vdash A, B}\,w}{\vdash A, A \to B}\,{\to}\text{-}r \qquad \dfrac{}{A \vdash A}\,\textsf{id}}{\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\,{\to}\text{-}r}\,{\to}\text{-}l
$$

A cut-free proof of the Drinker's Paradox:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D(v) \vdash D(v)}\,\textsf{id}}{D(u), D(v) \vdash D(v), \forall y.D(y)}\,w}{D(u) \vdash D(v), D(v) \to \forall y.D(y)}\,{\to}\text{-}r}{D(u) \vdash D(v), \exists x.(D(x) \to \forall y.D(y))}\,\exists\text{-}r}{D(u) \vdash \forall x.D(x), \exists x.(D(x) \to \forall y.D(y))}\,\forall\text{-}r}{\vdash D(u) \to \forall y.D(y), \exists x.(D(x) \to \forall y.D(y))}\,{\to}\text{-}r}{\vdash \exists x.(D(x) \to \forall y.D(y)), \exists x.(D(x) \to \forall y.D(y))}\,\exists\text{-}r
$$

$$
\dfrac{\vdash \exists x.(D(x) \to \forall y.D(y)), \exists x.(D(x) \to \forall y.D(y))}{\vdash \exists x.(D(x) \to \forall y.D(y))}\,c\text{-}r
$$

Outline

Perspectives

Cut-elimination is arguably one the most important results in all of logic.

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Its applications are numerous:

- Conservativity results. (mathematical logic)
- Automating proof search. (logic programming)
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Theorem (Statman ’79, Orevkov ’82)
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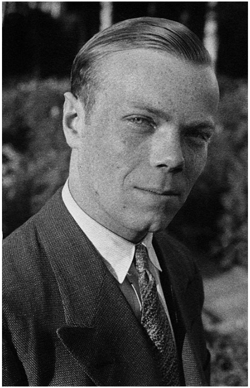
…and, of course, first-order logic remains undecidable.

Herbrand's theorem



Theorem (Herbrand's theorem)

If $\vDash \exists x.A(x)$ *then there is a finite set of terms* $\{t_i\}_{i=1}^{n}$ *s.t.* $\vDash A(t_1) \vee \cdots \vee A(t_n)$.

Proof idea of Herbrand's Theorem

The basic idea is that the terms in the Herbrand disjunction are just those from which the $\exists$ originates.

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The interesting cases are:

$$\exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, \exists x.A, \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

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Exercise: Try to check how each of the other cases works for yourself!

Digression: recovering richer expressivity

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All of our propositional rules are cut-free-invertible.

Proof idea.

By induction on the structure of the proof. Essentially we 'replace' each formula with its 'inverses'. □

Rules for $\vee$, $\wedge$ and an encoding in $\{\to, \bot\}$

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Recall that we can encode $\vee$ and $\wedge$ in $\{\rightarrow, \bot\}$ by adequacy:

$$
\begin{aligned}
A \vee B &\equiv \neg A \rightarrow B \\
A \wedge B &\equiv \neg(A \rightarrow \neg B)
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We could also have native rules for $\vee$ and $\wedge$:

$$\wedge\text{-}l\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\text{-}r\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee\text{-}r\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

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$$

$$
\vee\text{-}r\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}
$$

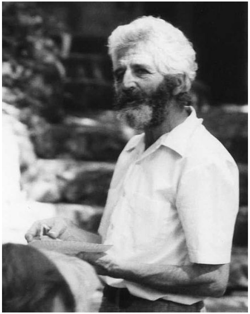
Proposition

The sequent system with $\vee$ and $\wedge$ native is also cut-free complete.

Proof idea.

Use cut-free-invertibility of the $\to$ and $\bot$ rules to simulate the rules above. □

Craig interpolation



Theorem (Craig Interpolation)

If $\vDash A \to B$*, there is some* I *in the common language of* A *and* B *s.t.* $\vDash A \to I$ *and* $\vDash I \to B$*.*

Proof of Craig Interpolation

By the previous arguments, we will work in the calculus including only the connectives $\neg$, $\vee$, $\wedge$ and all formulae in negation normal form, i.e. with negation only on atoms.

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Proof idea of Craig Interpolation.

Structural induction on a cut-free proof. For instance, if the proof ends with,

$$\wedge\text{-}r\ \dfrac{\overset{\pi_0}{\Gamma \vdash \Delta, A} \qquad \overset{\pi_1}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, A \wedge B}$$

and I_0, I_1 are interpolants obtained from π_0, π_1 resp., then we may define the new interpolant as $I_0 \wedge I_1$. □

Outline

The system LK

Initial and structural rules

$$\textsf{id}\ \frac{}{A \vdash A} \qquad w\text{-}l\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad w\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Propositional rules:

$$\bot\text{-}l\ \frac{}{\bot \vdash} \qquad \left(\bot\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\right) \qquad \to\text{-}l\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

Quantifier rules:

$$\forall\text{-}l\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

Cut rule:

$$cut\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

1. We saw that modus ponens could be simulated in LK. Give cut-free LK proofs for each of the three propositional axioms of $\mathcal{F}$:

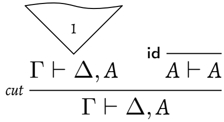
$$\begin{array}{rl} (wk) & A \to (B \to A) \\ (dist) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ (neg) & ((A \to \bot) \to \bot) \to A \end{array}$$

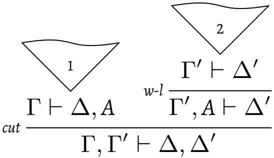
2. What about the quantifier axioms:

$$\begin{array}{l} \forall x.A \to A[t/x] \\ \forall x.(A \to B) \to (A \to \forall x.B) \quad \text{as long as } x \notin \mathsf{FV}(A) \end{array}$$

(**NB:** this concludes the proof of completeness of LK).

3. What is the Herbrand disjunction for the Drinker's Paradox?
4. How can we eliminate the cuts in the following cases?





Solutions to Exercise 1

$$(wk) \quad : \quad {\to}\text{-}r\,\dfrac{{\to}\text{-}r\,\dfrac{w\,\dfrac{\mathsf{id}\,\dfrac{}{A \vdash A}}{A, B \vdash A}}{A \vdash B \to A}}{\vdash A \to (B \to A)}$$

$$(neg) \quad : \quad {\to}\text{-}r\,\dfrac{{\to}\text{-}l\,\dfrac{{\to}\text{-}r\,\dfrac{w\text{-}r\,\dfrac{\mathsf{id}\,\dfrac{}{A \vdash A}}{A \vdash A, \bot}}{\vdash A, A \to \bot} \qquad w\,\dfrac{\bot\text{-}l\,\dfrac{}{\bot \vdash}}{\bot \vdash A}}{(A \to \bot) \to \bot \vdash A}}{\vdash ((A \to \bot) \to \bot) \to A}$$

$$(dist) \quad : \quad 3\cdot{\to}\text{-}r\,\dfrac{{\to}\text{-}l\,\dfrac{w\,\dfrac{\mathsf{id}\,\dfrac{}{A \vdash A}}{A \to (B \to C), A \vdash A, C} \qquad {\to}\text{-}l\,\dfrac{w\,\dfrac{\mathsf{id}\,\dfrac{}{A \vdash A}}{B, A \vdash A, C} \qquad {\to}\text{-}l\,\dfrac{w\,\dfrac{\mathsf{id}\,\dfrac{}{B \vdash B}}{B, A \vdash B, C} \qquad w\,\dfrac{\mathsf{id}\,\dfrac{}{C \vdash C}}{C, B, A \vdash C}}{B \to C, B, A \vdash C}}{A \to (B \to C), B, A \vdash C}}{A \to (B \to C), A \to B, A \vdash C}}{\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))}$$

Solutions to Exercise 2

$$
\dfrac{\dfrac{\dfrac{}{A[t/x] \vdash A[t/x]}\ \textsf{id}}{\forall x.A \vdash A[t/x]}\ \forall\text{-}l}{\vdash \forall x.A \to A[t/x]}\ \to\text{-}r
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \textsf{id}}{A \vdash B, A}\ w\text{-}r \qquad \dfrac{\dfrac{}{B \vdash B}\ \textsf{id}}{B, A \vdash B}\ w\text{-}l}{A \to B, A \vdash B}\ \to\text{-}l}{\forall x.(A \to B), A \vdash B}\ \forall\text{-}l}{\forall x.(A \to B), A \vdash \forall x.B}\ \forall\text{-}r \quad x \notin \mathsf{FV}(\forall x.(A \to B), A)}{\forall x.(A \to B) \vdash A \to \forall x.B}\ \to\text{-}r}{\vdash \forall x.(A \to B) \to (A \to \forall x.B)}\ \to\text{-}r
$$

Outline

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