Introduction to Proof Theory

Lectures 5 & 6 - Some applications of proof theory

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These slides are available at `http://www.anupamdas.com/wp/lss18/`.

Outline

If something exists, can we find it?

Suppose that we are working in some first-order theory S, and we have a proof

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We have established that there exists *some* object satisfying A. But can we actually exhibit one?

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The search for witnesses for existential statements is theme which lies at the heart of proof theory.

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But how do we actually find our oldest person?

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Moreover, each of them are proven using very simple reasoning - and taking as an axiom some reasonable upper bound on the age of humans!

Key point. Existential statements are very strong! It can be very difficult to find an actual witness.

Can we make this idea more formal?

Which reasoning principle is the culprit?

The difficulty in producing witnesses for existential statements is primarily due to the negation axiom:

$$\neg\neg A \rightarrow A$$

which is equivalent to the statement

$$\neg A \vee A$$

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A typical use of excluded-middle in the proof of an existential statement would be

- If A then $P(t_1)$,
- If $\neg A$ then $P(t_2)$,
- Therefore since $\neg A \vee A$ then $\exists x.P(x)$.

But we don't know which of t_1 or t_2 works!

Rational powers

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Otherwise, $\sqrt{2}^{\sqrt{2}}$ must be irrational, and we can set $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, since

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While the above proof gives us two *candidates* for x and y, namely

$$(x, y) = (\sqrt{2}, \sqrt{2}) \text{ or } (\sqrt{2}^{\sqrt{2}}, \sqrt{2})$$

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Remark. Actually, it is known that $\sqrt{2}^{\sqrt{2}}$ is irrational, but this is a deep result in its own right.

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FUNDAMENTAL THEOREM OF ARITHMETIC: every number has a prime factorisation.

Proof (Aristotle, Euclid).

Suppose there are only finitely many primes and label them $p_1, \ldots, p_k$. Apply the fundamental theorem to $p_1 \cdot \cdots \cdot p_k + 1$ to find a prime factor. This cannot be any of the p_i. □

What constructive information can we extract from this proof?

- The fundamental theorem of arithmetic gives us a factoring algorithm.
- Given primes $p_1, \ldots, p_k$, we simply factor $p_1 \cdots p_k + 1$ to find a new prime.

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We also derive a bound: for any number n there is a prime p with $n < p \leq n! + 1$.

Outline

1. What does it mean to exist?
2. A less basic witnessing theorem
3. ‘Intuitionistic’ reasoning
4. From semantics to syntax: the scalability of Gentzen
5. From richer semantics to a newer proof theory
6. Personal perspectives: a revolution in proof theory
7. References

Peano Arithmetic, in sequent style

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We give a sequent-style system for it by adding to LK the following initial sequents,

- $0 = s(a) \vdash$
- $\mathsf{s}(a) = \mathsf{s}(b) \vdash a = b$
- $\vdash a + 0 = a$
- $\vdash a + \mathsf{s}(b) = \mathsf{s}(a + b)$
- $\vdash a \times 0 = 0$
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and, for each formula A, a corresponding induction rule:

$$\frac{\Gamma \vdash \Delta, A(0) \quad \Gamma, A(a) \vdash \Delta, A(\mathsf{s}a)}{\Gamma \vdash \Delta, A(t)}\ ind \quad (a \notin \mathsf{FV}(\Gamma, \Delta, A))$$

Existential proofs and a normal form

Convention: all formulae are over $\{\neg, \vee, \wedge\}$ in De Morgan Normal form (i.e. negation only on atoms). We also close all rules by De Morgan duality, so that there is no changing of sides in a proof.

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Proposition ('Free-cut' elimination)

Any $I\Sigma_1'$-provable sequent of only existential formulae can be proved using only existential formulae.

Existential proofs and primitive recursion

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Recall the primitive recursive functions that Michael showed:

Definition

The **primitive recursive** functions is the smallest class of functions containing $\mathsf{0}, \mathsf{s}, +, \times$, projections and closed under composition and primitive recursion: if g, h are primitive recursive then so is f defined as follows:

$$\begin{array}{rcl} f(\mathsf{0}, \vec{x}) & = & g(\vec{x}) \\ f(\mathsf{s}x, \vec{x}) & = & h(x, \vec{x}, f(x, \vec{x})) \end{array}$$

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The main result we will show is:

Theorem (Parsons '72)

If $I\Sigma_1' \vdash \forall \vec{x}. \exists ! y. A(x, y)$, where A is quantifier-free, then $A(\vec{x}, y)$ is the graph of a primitive recursive function.

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MOTTO: *induction on semi-recursive predicates captures precisely primitive recursion.*

(Dirk will say much more about this kind of stuff next week!)

Some bootstrapping

The following functions are primitive recursive:

<u>CONDITIONAL</u>

The function:

$$\mathsf{cond}(x, y, z) := \begin{cases} y & x = 0 \\ z & \text{otherwise} \end{cases}$$

NB: think of this as a $\texttt{if} - \texttt{then} - \texttt{else}$ construction.

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For each quantifier-free formula $A(\vec{a})$ (with all free variables displayed), the function:

$$f_A(\vec{x}) := \begin{cases} 1 & \mathbb{N} \vDash A(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

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EXERCISE: Prove that these functions are primitive recursive for yourself.

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We will show that, for any provable existential sequent,

$$\exists x_1.A_1, \ldots, \exists x_m.A_m \vdash \exists y_1.B_1, \ldots, \exists y_n.B_n$$

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How should we prove this? Let us proceed by structural induction…

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Define $f_j(\vec{a}, \vec{x}, x) := f_j'(x, \vec{a}, \vec{x})$.

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What about the universal quantifier or negation cases, which could be non-constructive? There are none by our normal form!

Structural rules and induction

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Then define:

$$f_{n+1}(\vec{a}, \vec{x}) := \begin{cases} f_{n+1}'(\vec{a}, \vec{x}) & \mathbb{N} \models A[f_{n+1}'(\vec{a}, \vec{x})/x] \\ f_{n+2}'(\vec{a}, \vec{x}) & \text{otherwise} \end{cases}$$

The induction case

The most interesting case is induction:

$$\text{ind}\ \frac{\Gamma \vdash \Delta, \exists x.A(\mathsf{0}) \quad \Gamma, \exists x.A(a) \vdash \Delta, \exists x.A(\mathsf{s}a)}{\Gamma \vdash \Delta, \exists x.A(t)}$$

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Now we may simply set $f(\vec{a}, \vec{x}) := f'(t, \vec{a}, \vec{x})$.

Outline

A problem with pure reasoning

We are in a strange situation: Perfectly valid reasoning allows us to conclude

$$\exists x, y \notin \mathbb{Q}.\ x^y \in \mathbb{Q}$$

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The question of what it means for an object to exist, or for a statement to be true, created a famous divide in the logical community, and led to the development of **intuitionism**.

Formalism vs Intuitionism

<u>Formalism</u> (led by D. Hilbert, see [Weir, 2015])

- Based on *syntax*.
- Mathematics is a game of symbols: something 'exists' if it can be derived from mathematical axioms by logical inference rules.
- In particular, if we can show the nonexistence of an object is false, then the object must exist.

<u>Intuitionism</u> (led by L. E. J. Brouwer, see [Iemhoff, 2016])

- Based on *semantics*.
- Mathematics is a mental construction: Something exists only if it can be exhibited.
- In particular, if we can show the nonexistence of an object is false, that doesn't necessarily mean that the object exists!

Combining intuitionism with formalism

So far in this course we have broadly taken the formalist approach. However, we can adapt our deductive systems so that they aligns with the principle of intuitionism. As we already mentioned, the problem lies with the negation axiom, so why not just remove it?

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Definition (A Hilbert-Frege system for intuitionistic first-order logic)

Take the system we defined in Lecture 1, but replace the axiom schema

$$\neg\neg A \to A$$

with the axiom **ex falso quodlibet**:

$$\bot \to A$$

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We have $\Gamma \vdash_i A \Rightarrow \Gamma \vdash A$, but not conversely. For example,

$$\nvdash_i A \vee \neg A$$
$$\nvdash_i \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$$
$$\nvdash_i \exists x(P(x) \to \forall y.P(y))$$

An intuitionistic sequent calculus

The formulation of intuitionistic logic in the sequent calculus turns out to be incredibly simple, which highlights once again the elegance of this system.

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Theorem (Gentzen, '34)

Let LJ *denote the restriction of the system* LK *in which we are only allowed to have one formula on the right, i.e. only contains sequents of the form* $\Gamma \vdash A$*. Then* LJ *is sound and complete for intuitionistic logic.*

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This is a quite remarkable result: a fundamentally semantic and philosophical notion is reflected by a purely syntactic criterion.

We also get cut-elimination for free:

Observation

Cut-elimination for LK *applied to* LJ *proofs yields cut-free* LJ *proofs.*

Corollary

Intuitionistic propositional logic is decidable.

Observe how the single-formula-on-the-right condition is broken by LK proofs of fundamentally classical theorems:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\textsf{id}}{A \vdash A, \bot}\,w\text{-}r}{\vdash A, A \to \bot}\,{\to}\text{-}r \qquad \dfrac{\dfrac{}{\bot \vdash}\,\bot\text{-}l}{\bot \vdash A}\,w}{\dfrac{(A \to \bot) \to \bot \vdash A}{\vdash ((A \to \bot) \to \bot) \to A}\,{\to}\text{-}r}\,{\to}\text{-}l
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The existence property demonstrates that for intuitionistic logic, an object exists if and only if it can be constructed!

Outline

Adapting LK

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Let us consider a few case studies that you might have heard of…

Relevan(t+ce) logic

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For a concise introduction, consult:

- [Mares, 2014]

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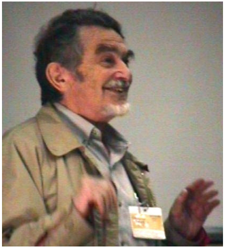
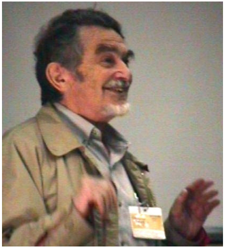
There is a lot more to say about *LL*. Here is a concise introduction:

- [Di Cosmo and Miller, 2016]

Lambek calculus



The **Lambek calculus** is a substructural logic obtained by combining resource-sensitivity with intuitionism.



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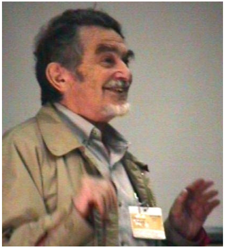

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- In mathematics, it is modelled by **residuated lattices**.
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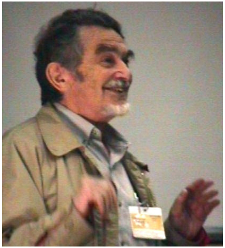

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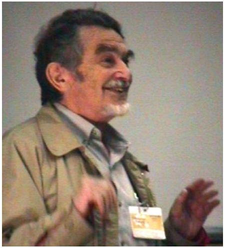

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For more information, consult:

- [Moortgat, 2014].

Outline

A richer semantics

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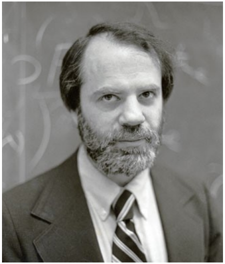

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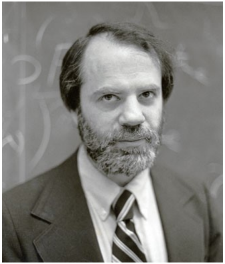

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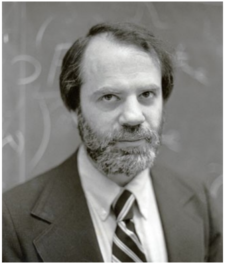

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This has lead to fundamental advances in *philosophy*, *mathematics* and *computer science*.

Modal logics

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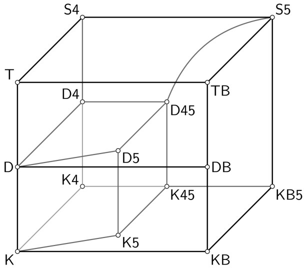

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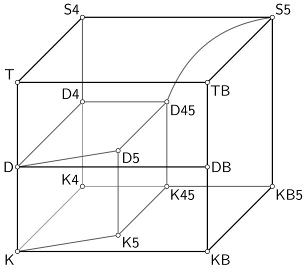

For a concise introduction, see:

- [Garson, 2016].

For a historical perspective, in particular contrasting the syntactic tradition, à la Lewis, and the semantic tradition, à la Kripke-Joyal, see:

- [Ballarin, 2017].

Gentzen calculi for modal logics

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Extensions such as $S4$ are similarly handled by adding:

$$t\,\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \qquad 4\,\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}$$

Remarkably, these calculi again enjoy cut-elimination:

Theorem

Cut-elimination holds for the sequent calculi for K and $S4$.

Corollary

- ***Interpolation*** *for modal logics.*
- ***Satisfiability solving*** *for modal logics.*

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Theorem (Mints, Pottinger, Avron, informally)

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Since this work, uniform and modular treatments have been found for all logics in the modal cube (and beyond!) in several other variations of sequent:

- Sequents are lists.
- Hypersequents are lists of lists.
- Nested sequents are trees.
- Labelled sequents are graphs.
- ...

Outline

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As we speak, there is a revolution under way towards a proof theory:

- with more structure. (*e.g.* (hyper + labelled + nested) sequents, cyclic proofs)
- that is more compositional. (*e.g.* deep inference, categorical logic, natural deduction, proof nets)
- that is more symmetric. (*e.g.*display calculus, deep inference)

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Let us look at two recent (very personally biased) developments…

Deep inference

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<u>Achievements include:</u>

- Modular proof-theoretic treatments of substructural and modal logics.
- Cut-elimination proofs, including finer extraction of interpolants and witnesses.
- Much shorter proofs!

Structure at the level of a proof

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A **cyclic proof** is one that allows cyclic reasoning. This can sometimes be meaningful!

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{b^2 = 2c^2 \vdash}\;\bullet}{c < a, 4c^2 = 2b^2 \vdash}}{\exists x < a. a = 2x, a^2 = 2b^2 \vdash}\;\bullet}{a^2 = 2b^2 \vdash}}{\vdash \forall x, y.\, x^2 \neq 2y^2}
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We now have proof theoretic treatments of:

- Fragments of the modal μ-calculus.
- Substructural logics with fixed points.
- First-order logic with inductive definitions
- Fragments of arithmetic.

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There are an increasing number of emerging applications in *computer science* and *mathematics*.

Thank you all!

Outline

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