Proof interpretations: a modern perspective

Lecture 5 - Proof interpretations today

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These slides are available at `http://www.anupamdas.com/nasslli18`.

Outline

The potential of program extraction

We have already seen some examples of witness extraction from $\forall\exists$ statements, our running example being

Theorem

There exists a function $X : \mathbb{N} \to \mathbb{N}$ such that for all n we have $X(n) \geq n$ and $X(n)$ prime.

But you don't need sophisticated proof theoretic techniques to be able to do this. So are there examples where the formal analysis of a proof can yield genuinely new numerical information from proofs?

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The answer is an emphatic YES. This is the so-called 'proof mining' program.

Central to the success of proof mining program are the following phenomena:

- One can typically extract a witnesses for $\forall\exists$ statements even when the underlying proofs are highly non-constructive;
- Certain mathematical principles, particularly forms of *compactness*, do not contribute to the complexity of extracted bounds, leading to surprisingly simple polynomial bounds from proofs which employ heavy machinery from analysis.

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- **2018: Where to next?**

Example: Uniqueness of best approximation

Theorem

Let $n \in \mathbb{N}$ and $f \in C[0,1]$ be fixed. Let

$$dist(f, P_n) := \inf_{p \in P_n} \|f - p\|$$

where P_n is the space of all polynomials with degree $\leq n$. Then there exists a polynomial of best approximation i.e. a polynomial p^* such that

$$\|f - p^*\| = dist(f, P_n),$$

and moreover, this polynomial is unique i.e. for all $p_1, p_2 \in P_n$

$$\bigwedge_{i=1,2} (\|f - p_i\| = dist(f, P_n)) \to p_1 = p_2.$$

A proof theoretic analysis of uniqueness

Let's look a bit more closely at uniqueness:

$$\forall n \in \mathbb{N} \forall f \in C[0,1] \forall p_1, p_2 \in P_n \left(\bigwedge_{i=1,2} (\|f - p_i\| = \mathrm{dist}(f, P_n)) \to p_1 = p_2 \right).$$

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Now, equality $=$ over the real numbers is actually a $\forall$-statement and so written out fully, uniqueness becomes

$$\begin{cases} \forall n \in \mathbb{N} \forall f \in C[0,1] \forall p_1, p_2 \in P_n \\ \left(\forall j \bigwedge_{i=1,2} (\|f - p_i\| - \operatorname{dist}(f, P_n) < 2^{-j}) \to \forall k \, \|p_1 - p_2\| < 2^{-k} \right). \end{cases}$$

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The (partial) functional interpretation of this is the following:

$$\begin{cases} \forall n, k \in \mathbb{N} \forall f \in C[0,1] \forall p_1, p_2 \in P_n \exists j \\ \left(\bigwedge_{i=1,2} (\|f - p_i\| - \operatorname{dist}(f, P_n) < 2^{-j}) \to \|p_1 - p_2\| < 2^{-k} \right). \end{cases}$$

A modulus of uniqueness

In the case of both the uniform norm and the L_1 norm, it is possible to extract a term Φ of System T such that

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Explicit moduli of uniqueness are given in the following papers:

- de La Vallée Poussin's proof of uniqueness of best Chebychev approximation [Kohlenbach, 1993a];
- Young's proof of uniqueness of best Chebychev approximation [Kohlenbach, 1993b];
- Cheney's proof of uniqueness of best L_1 approximation [Kohlenbach and Oliva, 2003a].

In some cases these results even improved known results in the literature.

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Potential new areas:

- Number theory?
- Probability theory?
- Financial mathematics?

Outline

The monotone convergence theorem

Recall in the last lecture we discussed the *monotone convergence principle*:

Theorem

Let (x_n) be a nondecreasing sequence of rational numbers in $[0,1]$. Then

$$\forall k \exists n \forall m (|x_{n+m} - x_n| \leq 2^{-k}).$$

The monotone convergence theorem

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$$\forall k \exists n \forall m (|x_{n+m} - x_n| \leq 2^{-k}).$$

We learned that in general there is no computable $N : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall k, \forall m (|x_{N(k)+m} - x_{N(k)}| \leq 2^{-k}),$$

due to the result of Specker.

But we now have a procedure for dealing with non-computable statements like this.

Let's first take a look at a proof.

Proving the monotone convergence theorem

Proof.

Suppose that the monotone convergence principle fails i.e. there exists some k such that

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Define the function $\tilde{g}(n) = n + g(n)$. Then we have a sequence

$$0 \leq x_0 < x_{\tilde{g}(0)} < x_{\tilde{g}(2)(0)} < \ldots <$$

with $x_{\tilde{g}^{(i+1)}(0)} - x_{\tilde{g}^{(i)}(0)} > 2^{-k}$, therefore

$$x_{\tilde{g}^{(2^k)}} > 1$$

a contradiction. □

The computational content of the proof

Our proof gave us some indirect computational information, namely

$$\forall k, g \exists n \leq \tilde{g}^{(2^k)}(0)(|x_{n+g(n)} - x_n| \leq 2^{-k}),$$

or in other words

$$\forall k, g \exists n \leq \tilde{g}^{(2^k)}(0) \forall i, j \in [n, n + g(n)](|x_i - x_j| \leq 2^{-k})$$

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Note that we can rephrase this statement entirely, so as only to refer to a finite part of (x_n). Let $M = \tilde{g}^{(2^k+1)}(0)$. We have the following:

Theorem (Finite convergence principle)

Let $k \in \mathbb{N}$, $g : \mathbb{N} \to \mathbb{N}$, and suppose that $0 \leq x_0 \leq x_1 \leq \ldots \leq x_M \leq 1$, where M is a sufficiently large number which depends only on k and g. Then there exists some $0 \leq n \leq n + g(n) \leq M$ such that $|x_i - x_j| \leq 2^{-k}$ for all $n \leq i, j \leq n + g(n)$.

This is the so-called *finite convergence principle*, made explicit by T. Tao's in

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Idea. Proof interpretations do much more that just extracting numerical information. They help us understand and formalize the connection between infinitary and finintary statements in mathematics.

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From function classes to arithmetics

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Write $\mathsf{I}\Sigma_1$ for the fragment of PA that only allows induction on $\exists$ formulae (*i.e.*, *semi-recursive* predicates). The following is a seminal result:

Theorem (C. Parsons '72)

If $\mathsf{I}\Sigma_1 \vdash \forall x.\exists! y.Q(x,y)$, *then* $Q(x,y)$ *is the graph of a primitive recursive function.*

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This is proved by using a restriction $\hat{\mathsf{T}}$ of system T to predicative recursors:

$$\begin{aligned} \hat{R}(f,g,0,b) &= f(b) \\ \hat{R}(f,g,\mathsf{s}n,b) &= g(n,\hat{R}(f,g,n,b),b) \end{aligned}$$

Indeed, this system allows us to recover a *bona fide* interpretation $\mathsf{I}\Sigma_1 \to \hat{\mathsf{T}}$ that satisfies all the usual properties we would expect.

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Indeed, this system allows us to recover a *bona fide* interpretation $\mathsf{I}\Sigma_1 \to \hat{\mathsf{T}}$ that satisfies all the usual properties we would expect.

Furthermore $\hat{\mathsf{T}}$ may naturally be interpreted, for appropriate theorems, into a quantifier-free classical system PRA over the primitive recursive functions.

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In fact, there is another way! It is known as the **Witness Function Method**, due to G. Mints and S. Buss.

Some important *differences*:

- Works directly in classical logic.
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NB: However the engines behind these results are rather different. In particular the witnessing is only possible in classical logic thanks to the cut-elimination theorem. This result, at the same time, 'includes' the work required in translating $\hat{\mathsf{T}}$ to PRA.

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The basic object of reasoning is a **sequent**:

$$A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n \quad \text{interpreted as} \quad \bigwedge_{i=1}^{m} A_i \Rightarrow \bigvee_{j=1}^{n} B_j$$

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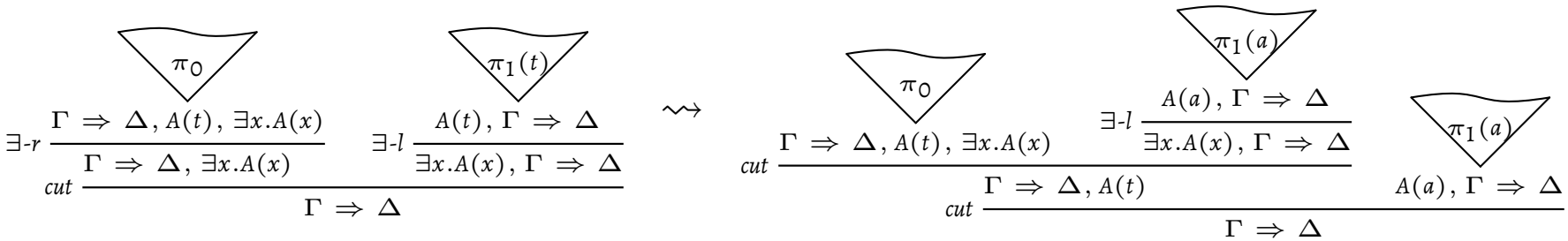
Hauptsatz (G. Gentzen, 1934)

For any first-order proof in the sequent calculus, there is one of the same conclusion s.t. all formulae occurring are subformulae of the conclusion, up to substitution of terms for variables.

NB: This is one of the deepest results in mathematical logic!

A glimpse of cut-reduction

A prototypical cut-reduction case:



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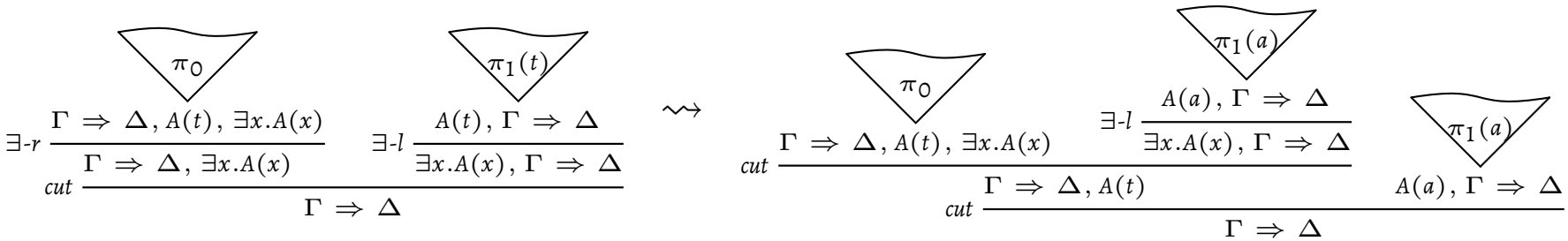

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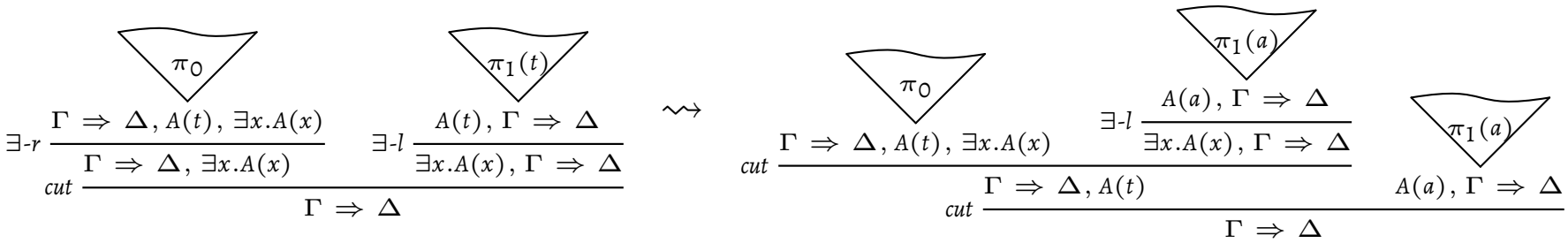

- Proved using a sophisticated induction measure.
- Non-elementary complexity.
- Closely related to computation via the *Curry-Howard correspondence*.

Here we rather use it for the following normal form property:

Corollary

In the case of arithmetic, any PA *proof can be written as a sequent-style proof where all formulae are subformulae of the conclusion, an induction formula or an axiom, up to…*

A rudimentary system for $I\Sigma_1$

For convenience, we may assume that all primitive recursive functions may occur as terms of $I\Sigma_1$. (Indeed, $I\Sigma_1$ can already prove that such functions are well-defined.)

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Instead of the N-interpretation, we have De Morgan normalisation of formulae and proofs:

Negation elimination

Every proof may be written in $\neg$-free form, via the following local translation on formulae:

$$\begin{array}{rcl} (\neg\, s \leq t)^n & := & t < s \\ (\neg(A \vee B))^n & := & \neg A^n \wedge \neg B^n \\ (\neg(A \wedge B))^n & := & \neg A^n \vee \neg B^n \end{array} \qquad \begin{array}{rcl} (\neg\exists x.A)^n & := & \forall x.\neg A^n \\ (\neg\forall x.A)^n & := & \exists x.\neg A^n \end{array}$$

No need for implication: this is classical logic!

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No need for implication: this is classical logic!
Along with cut-elimination we have:

Normalisation

Any $\mathrm{I}\Sigma_1$ proof of $\forall x.\exists y.Q(x,y)$ sentence can be translated into an arithmetic sequent proof of $\Rightarrow \exists y.Q(a,y)$ consisting of only $\exists$ formulae.

Interpretation into primitive recursive arithmetic.

PRA is something like system T restricted to only type 0 recursors. Equivalently, it is a quantifier-free version of PA equipped with terms for all primitive recursive functions. It is much simpler!

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Aside: We typically have to deal with formulae like $\forall x < t.\exists y.A(x,y,c)$. Again, the treatment of $\forall$ here differs from Dialectica, thanks to the **collection principle**:

$$\forall x < t.\exists y.A(x,y,c) \Rightarrow \exists z.\forall x < t.\exists y < z.A(x,y,c)$$

Interpretation of $\forall\exists$ theorems

Theorem (G. Mints '73, S. Buss '95)

For any $\mathsf{I}\Sigma_1$ *sequent proof of a sequent* $S(a)$ *there is a* PRA *proof of* $S_D(a, t(a))$ *for some appropriate term* t *of* PRA.

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Some of the key cases closely mirror those of Dialectica, e.g.:

$$\exists x.A(x,c) \vee \exists x.A(x,c) \Rightarrow \exists x.A(x,c) \quad \mapsto \quad A_d(a,c) \vee A_d(b,c) \to A_d(t(a,b,c),c)$$

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However, as well as implication, the $\forall$ also behaves differently:

$$\forall \frac{a < t(c) \Rightarrow \exists y.Q(a,y,c)}{\Rightarrow \exists z.\forall x < t(c).\exists y < z.Q(x,y,c)} \quad \mapsto \quad \frac{a < t(c) \Rightarrow Q(a,u(a,c),c)}{\Rightarrow \forall x < t(c).\exists y < v(c).Q(x,v(x,c),c)}$$

where $v(c) := \max\{u(a,c) : 0 < a < t(c)\}$.

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where $v(c) := \max\{u(a,c) : 0 < a < t(c)\}$.

Corollary

The $\mathsf{I}\Sigma_1 \vdash \forall x.\exists! y.Q(x,y)$ *just if* $Q(x,y)$ *is the graph of a primitive recursive function.*

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The approach has been extended to all of PA by S. Buss by extending by appealing to recursion on more complex orders.

However, the real advantage is for reasoning about weak theories, as we will see. In contrast the Dialectica interpretation is better for strong theories.

Key references

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  Philosophy of Science, 39(1):91–91

Outline

From logic to complexity

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Bounded arithmetic has been an immensely successful approach towards characterising *feasible* complexity classes. To date there exist characterisations of:

- Polynomial time (**P**) and each level of the polynomial hierarchy (**PH**).
- Feasible parallel time ($\mathbf{NC}^1$) and the $\mathbf{NC}^i$-$\mathbf{AC}^i$ hierarchy.
- Logarithmic space (**L**) and its (co-)nondeterministic versions ($\mathbf{NL} = \mathbf{coNL}$).
- Polynomial space (**PSPACE**), exponential time (**EXP**), ...

Key references

- Clote, P. and Kranakis, E. (2002). *Boolean Functions and Computation Models*
- Buss, S. R. (1986). *Bounded arithmetic.*
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For this we need a programming language for polynomial time!

Bounded recursion on notation (A. Cobham, 1964)

We say that f is defined by **bounded recursion on notation** from functions g, h, k if,

$$\begin{aligned} f(0) &= g \\ f(2x) &= h_0(x, f(x)) \quad \text{if } x \neq 0 \\ f(2x+1) &= h_1(x, f(x)) \end{aligned}$$

and $f(x) \leq k(x)$ for all x.

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Polynomial-time completeness: *The class of primitive recursive functions that only use recursion of the form above, plus the function $x^{\log y}$, are complete for polynomial time.*

Seminal results

We may define a theory S_2^1 that allows induction only on formulae of the form $\exists x < t.Q(x, a)$, analogous to $\mathsf{I}\Sigma_1$.

We may also define a quantifier-free theory PV based on bounded recursion on notation, analogous to PRA.

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Theorem (S. Buss, 1985, informally)

The $\forall\exists$ fragment of S_2^1 is interpreted by the WFM into PV. *In particular, if $S_2^1 \vdash \forall x.\exists!y.Q(x, y)$ then $Q(x, y)$ is the graph of a polynomial-time function.*

The case of polynomial time is particularly interesting since it serves as a litmus test for comparing different proof interpretations.

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Corollary (alternative to Buss, '85)

If intuitionistic-$S_2^1 \vdash \forall x.\exists y.A(x, y)$ for arbitrary A, then there is a term $t(x)$ of PV *such that intuitionistic-$S_2^1 \vdash \forall x.A(x, f(x))$.*

Applications to proof complexity

- Proof interpretations intimately connected to the size of proofs in propositional proof systems.
  (see, e.g., [Krajicek, 1995]).
- Bounded reverse mathematics: characterising the nondeterministic complexity of results in combinatorics and complexity.
  (see, e.g., [Cook and Nguyen, 2010])
- Elegant interplays between the WFM and realisability-style interpretations.
  [Das, 2016]
  Work in progress: proof complexity manipulations from the Dialectica interpretation.

Getting rid of bounds: implicit complexity

- Recursion-theoretic characterisations of complexity classes with no explicit resource bounds!
  (see, e.g., [Bellantoni and Cook, 1992]).
- Theories for these based on *ramification*, or *modalities*.
  (see, e.g., [Leivant, 1994])
- Resource sensitive logics + WFM = entirely implicit characterisations.
  [Baillot and Das, 2016]

Outline

The initial soundness theorem for the functional interpretation interprets Peano arithmetic in system T:

$$\mathrm{PA}^{\omega} \mapsto \mathrm{T}$$

Roughly speaking, PA^{ω} is just logic + induction, and the core of the soundness theorem is the following correspondence:

$$\text{induction} \mapsto \text{primitive recursion in all finite types}$$

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But many mathematical proofs go beyond Peano arithmetic, and in particular require some stronger principles such as comprehension, the axiom of choice, Zorn's lemma.

This gives rise to the following general problem:

$$\text{strong principle} \mapsto ???$$

Let's look at a simple version of the axiom of choice. Consider the drinkers paradox from the last lecture, now with an additional parameter n:

$$\forall n \exists x (P(n,x) \to \forall y P(n,y))$$

Then we can infer (using the axiom of choice) that there exists some function $f : \mathbb{N} \to X$ such that

$$\forall n (P(n, f(n)) \to \forall y P(n,y)).$$

> *For any countable set of pubs there is a countable set of drinkers such that if drinker n is drinking, then everyone in pub n is drinking.*

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> *For any countable set of pubs there is a countable set of drinkers such that if drinker n is drinking, then everyone in pub n is drinking.*

We already know how to solve the (classical) functional interpretation of ordinary drinkers paradox for pub n:

$$\forall n, g \exists x (P(n,x) \to P(n, g(x)))$$

namely via the functional

$$\Phi(n,g) := \begin{cases} 0 & \text{if } P(n, g(0)) \\ g(0) & \text{if } \neg P(n, g(0)) \end{cases}$$

Now consider the functional interpretation of our 'function' form of the drinkers paradox

$$\exists f \forall n (P(n, f(n)) \to \forall y P(n, y)).$$

Which is

$$\forall \omega, \varphi \exists f (P(\omega f, f(\omega f)) \to P(\omega f, \varphi f))$$

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Let $a : \mathbb{N} \to X$ be a partial function and let $\hat{a}$ be the extension of this function with some canonical element of type X. Define

$$\Psi(\omega, \varphi)(a) := \begin{cases} \hat{a} & \text{if } \omega(\hat{a}) \in \mathrm{dom}(a) \text{ or } P(\omega(\hat{a}), \varphi(\hat{a})) \\ \Psi(\omega, \varphi)(a[\omega(\hat{a}) \mapsto \varphi(\hat{a})]) & \text{otherwise.} \end{cases}$$

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This is a new form of recursion over *extensions of partial functions*.

It is actually a weak form of bar recursion. The study of bar recursive extensions of System T and their relationship to choice/comprehension has been an active area of research in the last decade or so.

Extensions of proof interpretations and strong forms of recursion

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- Spector, C. (1962). Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles in current intuitionistic mathematics.
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Outline

In lecture 3 we 'extracted' a list reversing program. Our input was a proof that

$$\forall a \exists b \mathrm{Rev}(a, b)$$

and the result was a term t of System T (i.e. a program) satisfying

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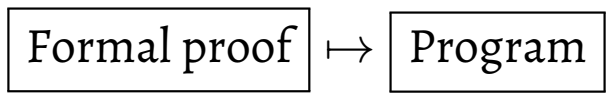
Can we automate proof interpretations i.e. write a piece of software which transforms a formal proof to a program:



In lecture 3 we 'extracted' a list reversing program. Our input was a proof that

$$\forall a \exists b \mathrm{Rev}(a, b)$$

and the result was a term t of System T (i.e. a program) satisfying

$$\forall a \mathrm{Rev}(a, t(a)).$$

Can we automate proof interpretations i.e. write a piece of software which transforms a formal proof to a program:

$$\boxed{\text{Formal proof}} \mapsto \boxed{\text{Program}}$$

Again, the answer is YES! There is even a proof assistant - MINLOG - dedicated to program extraction via functional interpretations:

http://www.mathematik.uni-muenchen.de/~logik/minlog/

What do we get out of formal extraction?

A program is written to satisfy a given specification:

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What a proof theorist might do.

- Write a formal proof that for any input, an output satisfying the specification exists.
- Push a button and extract a program.
- No debugging required!
- We may even get additional information, such as a bound on its complexity.

It's not quite as easy as that...

Sounds great! Why aren't programmers using proof interpretations?

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Sounds great! Why aren't programmers using proof interpretations?

- *"Is your list sorting program as good as one a human would write?"*
- *"We use C++. What good to us is a program written in System T?"*
- *"Your extracted program takes up ten pages of text. How does it even work?"*
- *"Group X can already do formal verification and have developed an extremely successful tool."*
- *"Could your technique for synthesising programs be easily used by someone working at the Guardian newspaper?"*

These are all valid points...

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Scale: Formal verification is a huge business and lots of sophisticated tools have already been developed. On top of this, most big proof assistants (Coq, NUPRL, etc) can extract programs from proofs. A small community in proof theory, dedicated to a particular style of program extraction, cannot possibly compete with this directly.

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Accessibility: Ultimately, methods for synthesising verified programs are only useful if they can be used by a non-specialist.

Some first steps

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