Introduction to Proof Theory

Exercises 1 - First-order logic: syntax and semantics

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Sheffield University (virtual)
12-16 April 2021

These slides are available at `http://www.anupamdas.com/mgs21`.
Based on slides from ESSLLI'18, prepared with Thomas Powell.

Propositional logic I

1. Show $\{A, B, A \to (B \to C)\} \vdash C$ and hence $\vdash (A \to B \to C) \to B \to A \to C$.

   NB: *there was a bracketing typo in the original version! Sorry, the second part should follow immediately from the first by deduction.*

   Answer. The first part follows simply by two *mp* steps:

$$\textit{mp}\,\dfrac{B \qquad \textit{mp}\,\dfrac{A \quad A \to B \to C}{B \to C}}{C}$$

   From here the second part follows by invoking the deduction theorem three times, once for each hypothesis above.

2. We can extend $\mathcal{F}$ to include conjunction $\wedge$ by adding the following axioms:

$$\begin{aligned} (pair) &\quad p \to q \to (p \wedge q) \\ (pl) &\quad (p \wedge q) \to p \\ (pr) &\quad (p \wedge q) \to q \end{aligned}$$

Show that this extended system proves: $(A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$

NB: $A \leftrightarrow B$ is an abbreviation for $(A \to B) \wedge (B \to A)$.

Answer. Let us first prove the two directions of the bi-implication individually. We have the following derivation,

$$\dfrac{\dfrac{A \wedge B \qquad \dfrac{}{(A \wedge B) \to B}\,pr}{B}\,mp \qquad \dfrac{\dfrac{A \wedge B \qquad \dfrac{}{(A \wedge B) \to A}\,pl}{A}\,mp \qquad A \to B \to C}{B \to C}\,mp}{C}\,mp$$

whence the left-right implication follows by invoking the deduction theorem.

Propositional logic III

We also have the derivation,

$$\textit{mp}\;\dfrac{2mp\;\dfrac{B \qquad A \qquad \textit{pair}\;\dfrac{}{A \to B \to (A \wedge B)}}{A \wedge B} \qquad (A \wedge B) \to C}{C}$$

whence the right-left implication follows by invoking the deduction theorem (DT).

Finally, note that from a proof of any two formulas one can construct a proof of their conjunction by using *modus ponens* with the *pair* axiom (see, e.g., left subderivation above).

3. We haven't yet used the axiom (*neg*)! Show that $\mathcal{F}$ proves:
   - a. $\neg A \to (A \to B)$
   - b. $(\neg B \to \neg A) \to (A \to B)$
   - c. $((A \to B) \to A) \to A$

   Answer. We again make significant use of the deduction theorem. We give the following 'meta-derivations' for reasoning about the $\vdash$ relation:

Propositional logic IV

a

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A, A, \neg B \vdash \bot}\,mp}{\neg A, A \vdash \neg\neg B}\,DT}{\neg A, A \vdash B}\,neg, mp}{\vdash \neg A \to A \to B}\,2{\cdot}DT$$

(Here DT = 'deduction theorem'; recall also that $\neg A$ is just $A \to \bot$, so from A and $\neg A$ we can infer $\bot$ by *mp*.)

b

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg B \to \neg A, A, \neg B \vdash \bot}\,2{\cdot}mp}{\neg B \to \neg A, A \vdash \neg\neg B}\,DT}{\neg B \to \neg A, A \vdash B}\,neg, mp}{\vdash (\neg B \to \neg A) \to A \to B}\,2{\cdot}DT$$

(Again, note that from $\neg B$ and $\neg B \to \neg A$ we can infer by *mp* $\neg A$, which is $A \to \bot$, so from A and *mp* again we can infer $\bot$.)

c

$$mp\;\dfrac{mp\;\dfrac{\dfrac{\text{Follows from (a)}}{\vdash \neg A \to A \to B}}{\neg A \vdash A \to B} \qquad DT\;\dfrac{2\cdot mp\;\dfrac{}{(A \to B) \to A, \neg A, A \to B \vdash \bot}}{(A \to B) \to A, \neg A \vdash (A \to B) \to \bot}}{\dfrac{(A \to B) \to A, \neg A \vdash \bot}{neg, mp, DT\;\dfrac{(A \to B) \to A \vdash A}{DT\;\vdash ((A \to B) \to A) \to A}}}$$

4. (**Hard**). Write $\mathcal{F}(\to)$ for the fragment of $\mathcal{F}$ without the *neg* axiom.
   - a. Notice that $\mathcal{F}(\to)$ still satisfies the Deduction theorem!
   - b. Show $\mathcal{F}(\to)$ is not equivalent to $\mathcal{F}$ over $\to$-only formulae. (Non-conservativity)
   - c. On the other hand, show that $\mathcal{F}(\to) + ((p \to q) \to p) \to p$ is equivalent to $\mathcal{F}$ over $\to$-only formulae.

Answer, sketched.

1. Note that we never used the *neg* axiom in the proof of the deduction theorem!
2. There are various ways to approach this. We may note that $\mathcal{F}(\to)$ is sound for *intuitionistic logic*, whence we can construct a Kripke countermodel for 'Peirce's law' $(((A \to B) \to A) \to A)$. We could also use the cut-free completeness result of Lecture 4 to show that there is no closed proof search strategy for Peirce's Law.

Propositional logic VI

3. Let π be a $\mathcal{F}$ proof containing just the (sub)formulas $\{A_i\}_{i<n}$. Now simply replace $\perp$ everywhere in π by the formula $C = \bigwedge_{i<n} A_i$. Any instance of the *neg* axiom becomes, say, $N = ((A \to C) \to C) \to A$. Since in particular $\vdash C \to A$, we have that any such N is a consequence of $((A \to C) \to A) \to A$, which is just an instance of Peirce's Law.

1. Suppose we have a language with at least one constant c and a unary predicate symbol P. Investigate the so-called drinkers paradox:

$$\mathrm{DP} := \exists x(P(x) \to \forall y.P(y))$$

In natural language, this is popularly captured by the statement:

In any pub there is a person such that if they are drinking, then everyone is drinking.

Is DP valid? If so, can you sketch a derivation?

Answer. We give a semantic argument (under the completeness theorem).

- Case 1: everyone in the pub is drinking. Then any person suffices to witness the existential.
- Case 2: at least one person, a, is not drinking. Then a suffices to witness the existential (by vacuous implication).

NB: We will see an explicit syntactic proof in the *sequent calculus* soon!

2. Show that $\{A \to B\} \vdash \neg\forall x \neg A \to B$ when $x \notin \mathsf{FV}(B)$. This corresponds to the $\exists$-rule

$$\frac{A \to B}{\exists x\, A \to B}$$

Clarification: given the current notion of $\mathcal{F}$-derivability, $\vdash$, the question is a little ill-posed, since the LHS of $\vdash$ should always be a set of sentences, so x cannot be free in A (or B). Try changing the hypothesis to $\forall x(A \to B)$ and assume there are no further free variables.

3. Outline a first-order theory whose models are the partial orders. Adapt this theory to characterise
   - total orders e.g. $\mathbb{Z}$ with $\leq$
   - total orders with a minimum element e.g. $\mathcal{N}$ with $\leq, 0$
   - partial orders with least upper bounds e.g. $\mathcal{P}(\mathcal{N})$ with $\subseteq$ and $\cup$

Answer. For the main question and the first bullet point, see the slides for Lecture 2. Existence of a minimum element is characterised by:

$$\exists x\, \forall y\, x \leq y$$

Existence of least upper bounds (of two elements) is characterised by:

$$\forall x, y\, \exists z\, (x \leq z \wedge y \leq \;\wedge \forall z' (x \leq z' \to y \leq z' \to z \leq z'))$$

Q: What about least upper bounds of arbitrary sets of elements? Can we express this in first-order logic?