INTRODUCTION TO PROOF THEORY

Exercises 2 - Metalogical foundations of first-order logic

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Exercises, Lecture 2 I

1. Show that the following forms of consistency are equivalent:

$$
\begin{array}{lcl}
(1) & : & \Gamma \nvdash \bot \\
(2) & : & \text{for no formula } A,\ \Gamma \vdash A \text{ and } \Gamma \vdash \neg A \\
(3) & : & \text{there is a formula } A \text{ s.t. } \Gamma \nvdash A
\end{array}
$$

Answer. We show $(1) \implies (2) \implies (3) \implies (1)$, all by contraposition.
$(1) \implies (2)$. Suppose indeed $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ for some formula A. Since $\neg A = A \to \bot$ we have, by (mp), that $\Gamma \vdash \bot$, as required.
$(2) \implies (3)$. Suppose $\Gamma \vdash A$ for all A. Then, in particular, $\Gamma \vdash \bot$ and $\Gamma \vdash \neg\bot$, as required.
$(3) \implies (1)$. Suppose $\Gamma \vdash \bot$. Then by *ex falso quodlibet*, $\bot \to A$, and (mp) we can derive any formula A from Γ. To justify *ex falso quodlibet*, here is a meta-derivation:

$$
\dfrac{\dfrac{\dfrac{}{\bot, \neg A \vdash \bot}}{\bot \vdash \neg\neg A}\,DT}{\dfrac{\bot \vdash A}{\vdash \bot \to A}\,DT}\,(mp), neg
$$

2. Show that the following formulas are equivalent,

$$\begin{array}{ll}(1) & \vdash ((A \to B) \to C) \to D \\ (2) & \vdash (A \to p) \to (p \to A) \to ((p \to B) \to C) \to D\end{array}$$

where p does not occur in any of A, B, C, D.

Hint: Under soundness and completeness, you may use either syntactic or semantic means, or a combination!

NB: there was an error in the original version, where the turnstile $\vdash$ was omitted in (1) and (2) above. Note that the two formulas are **not** *logically equivalent*, i.e. there is a model/assignment that satisfies one but not the other (**Q:** can you find it?).

Answer. Under soundness and completeness, we proceed semantically, i.e. we show that,

$$\begin{array}{ll}(1') & \models ((A \to B) \to C) \to D \\ (2') & \models (A \to p) \to (p \to A) \to ((p \to B) \to C) \to D\end{array}$$

are equivalent. We again proceed by contraposition.

$(1') \implies (2')$. Let $\mathcal{M}$ be a counter model of the formula of $(2')$, so we have $\mathcal{M} \vDash A \to p$ and $\mathcal{M} \vDash p \to A$ but $\mathcal{M} \nvDash ((p \to B) \to C) \to D$. Since the former tell us that p and A have the same valuation in $\mathcal{M}$, we can conclude from the latter that $\mathcal{M} \nvDash ((A \to B) \to C) \to D$, as required.

$(2') \implies (1')$. Let $\mathcal{M}$ be a countermodel of the formula of $(1')$ and extend $\mathcal{M}$ by assigning p the same Boolean value as A (recall that p does not occur in A, B, C, D, so we may do this). This forces the valuation of the formula of $(2')$ to be false too.

3. (**Hard**). A **graph** is a structure $G = (V, E)$ where $E \subseteq V \times V$. For $k \in \mathcal{N}$ we say that G is k-**colourable** if there is a 'colouring' function $c : V \to \{1, \ldots, k\}$ such that, whenever $(u, v) \in E$ we have $c(u) \neq c(v)$.

   Show, using the compactness theorem, that a (possibly infinite) graph is k-colourable if and only if every finite subgraph of it is k-colourable.

   Answer. Introduce propositional variables c_{vi} for each $v \in V$ and $i \in \{1, \ldots, k\}$. Let Γ be the theory consisting of the following formulas:

   a. (totality) $c_{v1} \vee \cdots \vee c_{vk}$, for each $v \in V$.
   b. (determinism) $\neg(c_{vi} \wedge c_{vj})$, for each $v \in V$ and $i, j \in \{1, \ldots, k\}$ with $i \neq j$.
   c. (colouring) $\neg(c_{ui} \wedge c_{vi})$, for each $(u, v) \in E$ and $i \in \{1, \ldots, k\}$.

Notice that any assignment of Boolean values to each c_{vi} satisfying Γ determines a k-colouring, and vice-versa.

Any finite subset $\Gamma' \subseteq \Gamma$ only mentions finitely many $v \in V$, and so is satisfied by some k-colouring of the induced finite subgraph, by assumption. Thus, by compactness, there is a k-colouring of G.