Introduction to Proof Theory

Exercises 3 - Gentzen's sequent calculus

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These slides are available at `http://www.anupamdas.com/mgs21`.

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Initial and structural rules

$$\text{id}\ \frac{}{A \vdash A} \qquad \textit{w-l}\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \textit{w-r}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \textit{c-l}\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \textit{c-r}\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Propositional rules:

$$\bot\textit{-l}\ \frac{}{\bot \vdash} \qquad \left(\bot\textit{-r}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\right) \qquad \to\textit{-l}\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\textit{-r}\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

Quantifier rules:

$$\forall\textit{-l}\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\textit{-r}\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\textit{-r}\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\textit{-l}\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

Cut rule:

$$\textit{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

1. We saw that modus ponens could be simulated in LK. Give cut-free LK proofs for each of the three propositional axioms of $\mathcal{F}$:

$$\begin{array}{rl} (wk) & A \to (B \to A) \\ (dist) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ (neg) & ((A \to \bot) \to \bot) \to A \end{array}$$

Answer.

$$\to\text{-}r\ \dfrac{\to\text{-}r\ \dfrac{w\ \dfrac{\mathsf{id}\ \dfrac{}{A \vdash A}}{A, B \vdash A}}{A \vdash B \to A}}{\vdash A \to (B \to A)}$$

$$\to\text{-}r\ \dfrac{\to\text{-}l\ \dfrac{\to\text{-}r\ \dfrac{w\text{-}r\ \dfrac{\mathsf{id}\ \dfrac{}{A \vdash A}}{A \vdash A, \bot}}{\vdash A, A \to \bot} \qquad w\ \dfrac{\bot\text{-}l\ \dfrac{}{\bot \vdash}}{\bot \vdash A}}{(A \to \bot) \to \bot \vdash A}}{\vdash ((A \to \bot) \to \bot) \to A}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{}
\LeftLabel{id}
\UnaryInfC{$A \vdash A$}
\doubleLine
\LeftLabel{w}
\UnaryInfC{$A \to (B \to C), A \vdash A, C$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$A \vdash A$}
\doubleLine
\LeftLabel{w}
\UnaryInfC{$B, A \vdash A, C$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$B \vdash B$}
\doubleLine
\LeftLabel{w}
\UnaryInfC{$B, A \vdash B, C$}
\AxiomC{}
\LeftLabel{id}
\UnaryInfC{$C \vdash C$}
\doubleLine
\LeftLabel{w}
\UnaryInfC{$C, B, A \vdash C$}
\LeftLabel{$\to$-l}
\BinaryInfC{$B \to C, B, A \vdash C$}
\LeftLabel{$\to$-l}
\BinaryInfC{$A \to (B \to C), B, A \vdash C$}
\LeftLabel{$\to$-l}
\BinaryInfC{$A \to (B \to C), A \to B, A \vdash C$}
\LeftLabel{$3 \cdot \to$-r}
\UnaryInfC{$\vdash (A \to (B \to C)) \to ((A \to B) \to (A \to C))$}
\end{prooftree}
$$

2. What about the quantifier axioms and rule:

$$\text{gen}\ \frac{A}{\forall x A} \qquad \begin{array}{l} \forall x\, A \to A[t/x] \\ \forall x\, (A \to B) \to (A \to \forall x\, B) \quad \text{as long as } x \notin \mathsf{FV}(A) \end{array}$$

(**NB:** this concludes the proof of completeness of LK).

Answer. Note that, if there is a $\mathcal{F}$-derivation $\Gamma \vdash A$, then Γ is a set of sentences so, in particular, $x \notin \mathsf{FV}(\Gamma)$. Therefore $\forall\text{-}r\ \dfrac{\Gamma \vdash A}{\Gamma \vdash \forall x A}$ is a correct inference step of LK.

$$\dfrac{\dfrac{\dfrac{}{A[t/x] \vdash A[t/x]}\ \mathsf{id}}{\forall x\, A \vdash A[t/x]}\ \forall\text{-}l}{\vdash \forall x\, A \to A[t/x]}\ \to\text{-}r$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ \mathsf{id} \quad \dfrac{}{B[y/x] \vdash B[y/x]}\ \mathsf{id}}{A \to B[y/x], A \vdash B[y/x]}\ \to\text{-}l}{\forall x\, (A \to B), A \vdash B[y/x]}\ \forall\text{-}l \quad x \notin \mathsf{FV}(A)}{\forall x\, (A \to B), A \vdash \forall x B}\ \forall\text{-}r \quad y \text{ fresh}}{\vdash \forall x\, (A \to B) \to A \to \forall x\, B}\ 2\cdot\to\text{-}r$$

3. What possible rules might we need if we had $\vee$ and $\wedge$ in our language?

   Answer. There are actually quite a few different options. Have a look at the Wikipedia article for the sequent calculus, and also linear logic to see some variations.

4. (**Puzzle**). There are 100 ants on a 1m stick, facing either end of the stick. They start moving at 1m/s and, every time they collide, they perfectly rebound. How long does it take for all the ants to fall off the stick (in the worst case)?

   Answer. Imagine each ant begins by carrying a baton, and that whenever they rebound they exchange batons. Now notice the following:

   a. At any moment each ant is carrying a baton.
   b. At any moment each baton is carried by an ant.
   c. The batons never change direction.

   By (b), the batons are always moving at 1m/s, and so by (c) each baton has fallen off the stick, in the worst case, in 1s. Finally, by (a), each ant has also fallen off the stick by 1s.

5. (**Still hard**). Revisiting Exercise 4 (Propositional logic) from Lecture 1: can you now show that Peirce's law, $((A \to B) \to A) \to A$, is not intuitionistically valid?

   Hint: use completeness of cut-free proof search in LJ for intuitionistic logic.

   Answer. The LJ proof search space for Peirce's law is cyclic (modulo contractions), so there cannot be an LJ proof of Peirce's law:

$$
\to\text{-}r\,\dfrac{
  \to\text{-}l\,\dfrac{
    \to\text{-}r\,\dfrac{
      \to\text{-}l\,\dfrac{
        \to\text{-}r\,\dfrac{
          \dfrac{\vdots}{(A \to B) \to A, A \vdash B}\,\bullet
        }{(A \to B) \to A, A \vdash A \to B}
        \qquad
        \text{id}\,\dfrac{}{A \vdash A}
      }{\underline{(A \to B) \to A}, A \vdash B}\,\bullet
    }{(A \to B) \to A \vdash A \to B}
    \qquad
    \text{id}\,\dfrac{}{A \vdash A}
  }{\underline{(A \to B) \to A} \vdash A}
}{\vdash ((A \to B) \to A) \to A}
$$