Introduction to Proof Theory

Exercises 3 - Gentzen's sequent calculus

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These slides are available at `http://www.anupamdas.com/mgs21`.

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The system LK

Initial and structural rules

$$\text{id}\ \frac{}{A \vdash A} \qquad \textit{w-l}\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \textit{w-r}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \textit{c-l}\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \textit{c-r}\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Propositional rules:

$$\bot\text{-}l\ \frac{}{\bot \vdash} \qquad \left(\bot\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\right) \qquad \to\text{-}l\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

Quantifier rules:

$$\forall\text{-}l\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

Cut rule:

$$\textit{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Exercises, Lecture 3

1. We saw that modus ponens could be simulated in LK. Give cut-free LK proofs for each of the three propositional axioms of $\mathcal{F}$:

$$\begin{array}{rl} (wk) & A \to (B \to A) \\ (dist) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ (neg) & ((A \to \bot) \to \bot) \to A \end{array}$$

2. What about the quantifier axioms and rule:

$$\text{gen}\ \frac{A}{\forall x A} \qquad \begin{array}{l} \forall x.A \to A[t/x] \\ \forall x.(A \to B) \to (A \to \forall x.B) \quad \text{as long as } x \notin \mathsf{FV}(A) \end{array}$$

   (**NB:** this concludes the proof of completeness of LK).

3. What possible rules might we need if we had $\vee$ and $\wedge$ in our language?

4. (**Puzzle**). There are 100 ants on a 1m stick, facing either end of the stick. They start moving at 1m/s and, every time they collide, they perfectly rebound. How long does it take for all the ants to fall off the stick (in the worst case)?

5. (**Still hard**). Revisiting Exercise 4 (Propositional logic) from Lecture 1: can you now show that Peirce's law, $((A \to B) \to A) \to A$, is not intuitionistically valid?

   Hint: use completeness of cut-free proof search in LJ for intuitionistic logic.