Introduction to Proof Theory

Exercises 4 - *Hauptsatz*: the cut-elimination theorem

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These slides are available at `http://www.anupamdas.com/mgs21`.

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Initial and structural rules

$$\textsf{id}\;\frac{}{A \vdash A} \qquad w\text{-}l\;\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad w\text{-}r\;\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\;\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad c\text{-}r\;\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Propositional rules:

$$\bot\text{-}l\;\frac{}{\bot \vdash} \qquad \bot\text{-}r\;\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \qquad \to\text{-}l\;\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\;\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

Quantifier rules:

$$\forall\text{-}l\;\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\;\frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\;\frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\;\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

Cut rule:

$$cut\;\frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

1. Consider the following key cut cases using $\wedge$ and $\vee$:

$$\mathit{cut}\;\dfrac{\vee\text{-}r\;\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta, A}}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\;\dfrac{\overset{\nabla_2}{\Gamma', A \vdash \Delta'} \quad \overset{\nabla_3}{\Gamma', B \vdash \Delta'}}{\Gamma', A \vee B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

$$\mathit{cut}\;\dfrac{\wedge\text{-}r\;\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta, A} \quad \overset{\nabla_2}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, A \wedge B} \qquad \wedge\text{-}l\;\dfrac{\overset{\nabla_3}{\Gamma', A, B \vdash \Delta'}}{\Gamma', A \wedge B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Can you define appropriate cut reductions for these situations?

Answer. The first derivation reduces to:

$$\mathit{cut}\;\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta, A} \quad \overset{\nabla_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

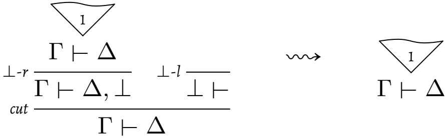
Exercises, Lecture 4 II

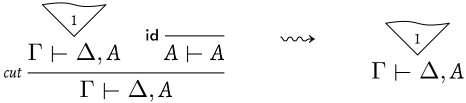
Note that we did not use the derivation (3). If (1) had ended with $\Gamma \vdash \Delta, B$ instead, we would have used (3) instead of (2).
The second derivation reduces to:

$$
c\,\dfrac{cut\,\dfrac{\dfrac{\overset{1}{\bigtriangledown}}{\Gamma \vdash \Delta, A} \qquad cut\,\dfrac{\dfrac{\overset{2}{\bigtriangledown}}{\Gamma \vdash \Delta, B} \quad \dfrac{\overset{3}{\bigtriangledown}}{\Gamma', A, B \vdash \Delta'}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

2. What about a key cut on $\bot$, or a cut where one subproof is just the identity axiom, $A \vdash A$?

Answer.





Note that, in both these cases, the cut is eliminated entirely. Thus these are the 'base cases' of the inductive arguent of cut-elimination.

3. Consider a very simple language consisting of just a function symbol not and a predicate symbol *True*, and let us extend LK by a single axiom,

$$\dfrac{}{\vdash True(x), True(\mathsf{not}(x))}\ LEM$$

What is the Herbrand disjunction of the following proof?

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash True(y), True(\mathsf{not}(y))}\ LEM}{\vdash True(y), \exists x\, True(x)}\ \exists\text{-}r}{\vdash \exists x\, True(x), \exists x\, True(x)}\ \exists\text{-}r}{\vdash \exists x\, True(x)}\ c\text{-}r$$

Answer. By inspection of the proof, the Herbrand set is $\{y, \mathsf{not}(y)\}$, so the Hebrand disjunction is $True(y) \vee True(\mathsf{not}(y))$.
NB: this question exemplifies how we can still extract Herbrand disjunctions for theories that extend LK by quantifier-free initial sequents (or Π_1 axioms).

Exercises, Lecture 4 V

4. Show that, if there is a proof of $\Gamma \vdash \Delta, A \to B$ of degree d, then there is a proof of $\Gamma, A \vdash \Delta, B$ of degree $\leqslant d$.
5. Show that, if there is a proof of $\Gamma, A \to B \vdash \Delta$ of degree d, then there are proofs of $\Gamma \vdash \Delta, A$ and $\Gamma, B \vdash \Delta$ of degree $\leqslant d$.
6. Conclude from (4) and (5) the following version of the critical lemma: Given a proof of degree d,

$$\textit{cut}\ \dfrac{\overset{P}{\Gamma \vdash \Delta, A \to B} \qquad \overset{Q}{\Gamma', A \to B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

where all cuts in P and Q have size $< |A \to B|$, there is a proof of $\Gamma, \Gamma' \vdash \Delta, \Delta'$ of degree $< d$.