INTRODUCTION TO PROOF THEORY

Exercises 4 - *Hauptsatz*: the cut-elimination theorem

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12 - 16 April 2021

These slides are available at `http://www.anupamdas.com/mgs21`.

Based on slides from ESSLLI'18, prepared with Thomas Powell.

Recap: the system LK

INITIAL AND STRUCTURAL RULES

$$\text{id}\ \frac{}{A \vdash A} \qquad w\text{-}l\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad w\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

PROPOSITIONAL RULES:

$$\bot\text{-}l\ \frac{}{\bot \vdash} \qquad \bot\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \qquad \to\text{-}l\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

QUANTIFIER RULES:

$$\forall\text{-}l\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

CUT RULE:

$$cut\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

1. Consider the following key cut cases using $\wedge$ and $\vee$:

$$\dfrac{\vee\text{-}r\;\dfrac{\overset{(1)}{\Gamma \vdash \Delta, A}}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\;\dfrac{\overset{(2)}{\Gamma', A \vdash \Delta'} \quad \overset{(3)}{\Gamma', B \vdash \Delta'}}{\Gamma', A \vee B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;\textit{cut}$$

$$\dfrac{\wedge\text{-}r\;\dfrac{\overset{(1)}{\Gamma \vdash \Delta, A} \quad \overset{(2)}{\Gamma \vdash \Delta, B}}{\Gamma \vdash \Delta, A \wedge B} \qquad \wedge\text{-}l\;\dfrac{\overset{(3)}{\Gamma', A, B \vdash \Delta'}}{\Gamma', A \wedge B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;\textit{cut}$$

Can you define appropriate cut reductions for these situations?

2. What about a key cut on $\bot$, or a cut where one subproof is just the identity axiom, $A \vdash A$?

3. Consider a very simple language consisting of just a function symbol not and a predicate symbol *True*, and let us extend LK by a single axiom,

$$\dfrac{}{\vdash \mathit{True}(x), \mathit{True}(\mathsf{not}(x))}\ \mathit{LEM}$$

What is the Herbrand disjunction of the following proof?

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathit{True}(y), \mathit{True}(\mathsf{not}(y))}\ \mathit{LEM}}{\vdash \mathit{True}(y), \exists x\, \mathit{True}(x)}\ \exists\text{-}r}{\vdash \exists x\, \mathit{True}(x), \exists x\, \mathit{True}(x)}\ \exists\text{-}r}{\vdash \exists x\, \mathit{True}(x)}\ c\text{-}r$$

4. Show that, if there is a proof of $\Gamma \vdash \Delta, A \to B$ of degree d, then there is a proof of $\Gamma, A \vdash \Delta, B$ of degree $\leqslant d$.
5. Show that, if there is a proof of $\Gamma, A \to B \vdash \Delta$ of degree d, then there are proofs of $\Gamma \vdash \Delta, A$ and $\Gamma, B \vdash \Delta$ of degree $\leqslant d$.

Exercises, Lecture 4 III

6. Conclude from (4) and (5) the following version of the critical lemma: Given a proof of degree d,

$$\textit{cut}\ \frac{\overset{P}{\Gamma \vdash \Delta, A \to B} \qquad \overset{Q}{\Gamma', A \to B \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

where all cuts in P and Q have size $< |A \to B|$, there is a proof of $\Gamma, \Gamma' \vdash \Delta, \Delta'$ of degree $\prec d$.