Introduction to Proof Theory

Lecture 2 - Metalogical foundations of first-order logic

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These slides are available at `http://www.anupamdas.com/mgs21`.

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Outline

Recap of system $\mathcal{F}$

$\exists x A :: \neg\forall x \neg A$
$A \vee B :: \neg A \to B$

Logical basis: $\{\bot, \to, \forall\}$, with $\neg A := A \to \bot$.

Definition (Axioms and rules of $\mathcal{F}$)

$\mathcal{F}$ has the following **axioms**:

(wk) $A \to (B \to A)$ ⇝ K combinator

(dist) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ ⇝ S combinator

(neg) $\neg\neg A \to A$

- $\forall x\, A \to A[t/x]$
- $\forall x(A \to B) \to A \to \forall x\, B$ where $x \notin \mathsf{FV}(A)$
- $\forall x(x = x)$
- $\forall x, y(x = y \to A \to A[y/x])$

$\Gamma \vdash A$

$\mathcal{F}$ also has two **inference rules**, namely:

$$\frac{A \quad A \to B}{B}\;mp \qquad\qquad \frac{A}{\forall x A}\;gen$$

A **structure** $\mathcal{D}$ consists of a *domain* D, along with suitably typed interpretations c_D, f_D, P_D, etc. of constant, function and relation symbols.

Definition (Valuation of formulas)

Valuation is a map $[-]^\sigma_D$: Form $\to \{0,1\}$ as follows:

$$[P(t_1,\ldots,t_k)]^\sigma_D := P_D([t_1]^\sigma_D,\ldots,[t_k]^\sigma_D)$$

$$[s=t]^\sigma_D := \begin{cases} 1 & \text{if } [s]^\sigma_D =_D [t]^\sigma_D \\ 0 & \text{otherwise} \end{cases}$$

$$[\bot]^\sigma_D := 0$$

$$[A \to B]^\sigma_D := [A]^\sigma_D \to [B]^\sigma_D$$

$$[\forall x.A]^\sigma_D := \begin{cases} 1 & \text{if for all } d \in D \text{ we have } [A]^{\sigma[x\mapsto d]}_D = 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, $\sigma[x \mapsto d]$ denotes the variable assignment which agrees with σ for all $y \neq x$, and maps x to d.

NB: valuation gives a reduction to simpler Boolean semantics, i.e. truth tables.

The relationship between $\vdash$ and $\to$

You may have noticed that $\vdash$ and $\to$ seem to have a similar meaning, although they correspond to two quite different things, namely:

$$A \vdash B : \textit{I can derive B if I am allowed to take A as a hypothesis}$$

$$\vdash A \to B : \textit{I can prove the implication } A \to B \textit{ without any hypotheses}$$

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Theorem (Deduction theorem)

Let A be a sentence. $\Gamma \vdash A \to B$ if and only if $\Gamma \cup \{A\} \vdash B$.

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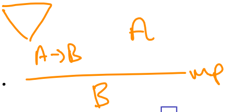
Proof (easy direction $\Rightarrow$).

Suppose that $(A_1, \ldots, A_n, A \rightarrow B)$ is a derivation of $A \rightarrow B$ from Γ.

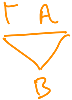


Then $(A_1, \ldots, A_n, A \rightarrow B, A, B)$ is a derivation of B from $\Gamma \cup \{A\}$. □

(Blank slide, if needed)

Proof (hard direction $\Leftarrow$)

Suppose that $(A_1, \ldots, A_n = B)$ is a derivation of B from $\Gamma \cup \{A\}$. We show by induction on i that $\Gamma \vdash A \to A_i$ for all $i = 1, \ldots, n$.



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If A_i is an axiom or element of Γ, then a derivation $\Gamma \vdash A \to A_i$ is given via

$$(mp)\,\frac{A_i \quad A_i \to (A \to A_i)}{A \to A_i}$$

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If A_i is A, then we have $\Gamma \vdash A \to A$ (we showed this earlier).

If A_i follows by (mp) from A_j and $A_j \to A_i$ where these occur previously in the derivation. By the induction hypothesis we have $\Gamma \vdash A \to A_j$ and $\Gamma \vdash A \to (A_j \to A_i)$, Suppose that $(A_1, \ldots, A_n = B)$ is a derivation of B from $\Gamma \cup \{A\}$. We show by induction on i that $\Gamma \vdash A \to A_i$ for all $i = 1, \ldots, n$.

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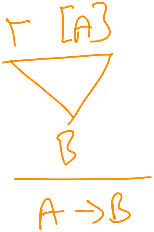
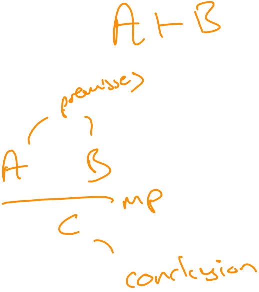
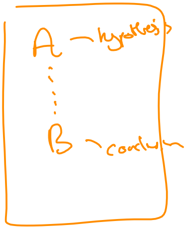
$$(mp)\ \dfrac{\dfrac{\overset{\text{IH}}{\vdots}}{A \to A_j} \qquad (mp)\ \dfrac{\dfrac{\overset{\text{IH}}{\vdots}}{A \to (A_j \to A_i)} \quad \dfrac{\text{dist}}{(A \to (A_j \to A_i)) \to (A \to A_j) \to (A \to A_i)}}{(A \to A_j) \to A \to A_i}}{A \to A_i}$$

is a derivation $\Gamma \vdash A \to A_i$.

Proof (hard direction ⇐), continued (blank slide)







Examples (blank slide)

Exercise 3 from yesterday

- $\vdash \neg A \to (A \to B)$

$$\neg A, A, \neg B \vdash \bot \quad \text{(mp)}$$

$$\neg A, A \vdash \neg\neg B \quad \text{(DT)}$$

$$\neg A, A \vdash B \quad \text{(mp, neg)}$$

$$\vdash \neg A \to A \to B \quad \text{(DT}^2\text{)}$$

$A \to \bot$ (for $\neg A$)

$\neg\neg B := \neg B \to \bot$

Outline

What's wrong with **Falso**?

Let me introduce you to the **Falso** proof system, that extends $\mathcal{F}$ by a single axiom:

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According to its website, `http://inutile.club/estatis/falso/`, **Falso** has remarkable properties:

	ZFC	Falso
Stronger than first-order arithmetic	Yes	Yes
Stronger than naive set theory	No	Yes
Number of axioms to memorize (less is better)	9	1
Maximal length required for proofs (less is better)	Unbounded	Less than a page
Proportion of true statements (more is better)	Less than 50%	Over 99.9%
Available 100% efficient theorem checkers	None	*Estatis Falso HyperVerifier*
Available 100% efficient proof assistants	None	*Estatis Falso HyperProver*
Resistant to Gödel attacks	No	Yes
Can prove its own consistency	No	Yes

Aside: On being *meaningful*

Proofs should be meaningful in a concrete sense. In particular, here we will insist that they only prove true things ('soundness').

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- $\mathcal{S}$ does not prove $\perp$.
- $\mathcal{S}$ does not prove a contradiction.
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In structural proof theory it is often the dynamics of a proof system which gives it meaning:

- Proof search in $\mathcal{S}$ corresponds to some computational process.
- There is a normalisation procedure for proofs in $\mathcal{S}$ that corresponds to a computational process.

$\rightsquigarrow$ 'proofs as programs'.

Proof implies truth

Recall that we write $\Gamma \vDash A$ if, for every assignment $\alpha : \mathsf{Prop} \to \{0, 1\}$, if for every $B \in \Gamma$ we have $\alpha(B) = 1$ then $\alpha(A) = 1$.

Theorem (Soundness)

If $\Gamma \vdash A$ *then* $\Gamma \vDash A$.

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We will prove this by structural induction on a derivation of A from Γ.

For the base cases, we must verify the axioms.
Here are truth tables for (wk) and (neg):

p	q	$q \to p$	$p \to (q \to p)$
0	0	1	1
0	1	0	1
1	0	1	1
1	1	1	1

p	$\neg p$	$\neg\neg p$	$\neg\neg p \to p$
0	1	0	1
1	0	1	1

Verifying the propositional axioms (continued)

For the axiom,

$$(\text{dist}) \quad : \quad ((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$$

let us write:

$$\begin{array}{lcl} A & : & q \to r \\ B & : & p \to A \end{array} \qquad \begin{array}{lcl} C & : & p \to q \\ D & : & p \to r \end{array}$$

(so (dist) is $B \to (C \to D)$)

p	q	r	A	B	C	D	$C \to D$	$B \to (C \to D)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1

Verifying the quantifier axioms (blank slide)

Show $[\forall x A \to A[t/x]]^{\sigma}_{D} = 1$

$[\forall x A \to A[t/x]]^{\sigma}_{D} = [\forall x A]^{\sigma}_{D} \to [A[t/x]]^{\sigma}_{D}$

Case 1: $[\forall x A]^{\sigma}_{D} = 0$ ✓

Case : $[\forall x A]^{\sigma}_{D} = 1$, i.e. for every $d \in D$, $[A]^{\sigma[x \mapsto d]}_{D} = 1$

Let $d = [t]^{\sigma}_{D}$.

$[A[t/x]]^{\sigma}_{D} = [A]^{\sigma[x \mapsto d]}_{D} = 1$ ✓

Inductive step

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Proposition (Verifying *modus ponens*)

If A and $A \to B$ are valid, then so is B.

Proof.

Let α be an assignment. We have,

$$
\begin{aligned}
1 &= \alpha(A \to B) && \text{by assumption} \\
&= \alpha(A) \to \alpha(B) && \text{by definition} \\
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Corollary (Consistency)

There is no formula A such that $\vdash A$ and $\vdash \neg A$.

Verifying generalisation (blank slide)

Outline

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Consider the sound system consisting of the following single axiom and no rules:

$$\frac{}{p \to p}$$

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Proviso: proofs should give us nontrivial insight.

Proofs are nondeterministic algorithms, so they should admit the ability to make 'clever choices'.

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NB: For completeness, I will focus on the setting of ***propositional logic***.

Reformulating completeness

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Typically, when proving metalogical results like completeness, we use the contrapositive formulation,

$\Gamma \models A \Rightarrow \Gamma \vdash A$

$\Leftrightarrow$

$$\Gamma \nvdash A \implies \Gamma \nvDash A$$

which, by (*neg*) and deduction, is equivalent to:

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Exercise: Show that this reformulation, conversely, implies completeness.

Maximally consistent sets

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Define $\Gamma_0 := \Gamma$ and inductively define Γ_{i+1} as follows:

$$\Gamma_{i+1} := \begin{cases} \Gamma_i \cup \{A_i\} & \text{if } \Gamma_i \vdash A_i \\ \Gamma_i \cup \{\neg A_i\} & \text{otherwise} \end{cases}$$

for uncountable languages "enumeration" becomes an instance of WO

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The point is to always define Γ_i such that it is consistent....

Consistency in the Lindenbaum construction

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Proof.

By induction on i.
For the base case we have $\Gamma_0 = \Gamma$ which is consistent by assumption. For the inductive step, suppose Γ_i is consistent.

- If $\Gamma_i \vdash A_i$ then $\Gamma_{i+1} = \Gamma_i \cup \{A_i\}$. So if Γ_{i+1} is inconsistent, *i.e.* $\Gamma_i, A_i \vdash \bot$, then $\Gamma_i \vdash \neg A_i$ by deduction, implying that Γ_i is already inconsistent.
- If $\Gamma_i \nvdash A_i$ then $\Gamma_{i+1} = \Gamma_i \cup \{\neg A_i\}$. So if Γ_{i+1} is inconsistent, *i.e.* $\Gamma_i, \neg A_i \vdash \bot$, then $\Gamma_i \vdash \neg\neg A_i$ by deduction, and so $\Gamma_i \vdash A_i$ by (*neg*). □

Limits of consistent chains

$\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots$

A consequence of the notion of proof is that consistency is preserved at limits:

Proposition

If $(\Delta_i)_{i \in \mathcal{N}}$ *is a consistent sequence of sets of formulae s.t.* $\Delta_0 \subseteq \Delta_1 \subseteq \cdots$*, then* $\Delta := \bigcup_{i \in \mathcal{N}} \Delta_i$ *is also consistent.*

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Proof sketch.

Since proofs are finite objects, only a finite number of hypotheses from Δ may be used in proofs of A and $\neg A$, which means that any contradiction would already be derivable from Δ_i for some $i \in \mathcal{N}$. □

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We have

$$\Gamma' \supseteq \Gamma_{k+1} = \begin{cases} \Gamma_k \cup \{A\} & \text{if } \Gamma_k \vdash A \\ \Gamma_k \cup \{\neg A\} & \text{otherwise} \end{cases}$$

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- If $\Gamma_k \vdash A$ then by definition $A \in \Gamma_{k+1} \subseteq \Gamma'$, contradiction.
- If $\Gamma_k \nvdash A$ then by definition $\neg A \in \Gamma_{k+1} \subseteq \Gamma'$. But then $\Gamma' \vdash \neg A$ and so $\Gamma' \cup \{A\}$ is not consistent, again a contradiction.

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This completes the proof of Lindenbaum's lemma.

Completeness from Lindenbaum

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Recall that this gives us the completeness result:

Theorem (Completeness, again)

If $\Gamma \vDash A$ then $\Gamma \vdash A$.

Reflection on the generality of this method

The completeness method we outlined here was, to be honest, a little overkill for classical propositional logic.

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However it is striking in its generality. Similar methods work for:

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However it is striking in its generality. Similar methods work for:

- First-order logic with respect to Henkin models.
- Modal logic with respect to frame properties.
- Intuitionistic and intermediate logics with respect to frame properties.

In all these cases, it is usually the construction of a ‘model’ from a maximally consistent set that is difficult.

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- The reduction of completeness to (consistency $\implies$ satisfiability) goes through as before, with the first-order notion $\models$ of semantic entailment.
- The Lindenbaum construction goes through as before for closed formulae.
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Proposition (Term models)

Let Γ be a maximally consistent set and write $s \approx t$ if $\Gamma \vdash s = t$. There is a model whose domain is $\mathsf{Ter}/\approx$, and whose true formulae are precisely those of Γ.

NB: For details about Henkin's construction, consult [Smullyan, 1968].

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In a formal sense, the statement of completeness itself for first-order logic is much more complex than that for propositional logic:

Theorem (Metalogical strength of completeness)

Over a weak base theory, completeness for FOL (over countable languages) is equivalent to Weak König's Lemma.

Outline

A curious corollary of completeness...

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The following result is both simple and remarkably useful:

We say that a set Γ is **finitely satisfiable** if every finite subset $\Gamma' \subseteq \Gamma$ is satisfiable.

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Example format of an application of compactness:

- For each $n \in \mathcal{N}$ it is possible to toss a coin n times and get heads each time.
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NB: In the propositional case, this is just an instance of the Tychonoff compactness theorem from topology.

Partial orders

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Definition (Orders)

A **partial order** is a structure $(X, \leq)$ such that, for all $x, y, z \in X$:

- $x \leq x$.
- $x \leq y, y \leq x \implies x = y$.
- $x \leq y, y \leq z \implies x \leq z$.

If, furthermore, for all $x, y \in X$ either $x \leq y$ or $y \leq x$, then we say that $(X, \leq)$ is a **total order**.

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Example

- $(\mathcal{N}, \leq)$ is a total order with a minimum element, 0.
- $(\mathbb{R}, \leq)$ is a total order that is dense: whenever $x < y$ there is z s.t. $x < y < z$.
- $(\mathcal{P}(\mathcal{N}), \subseteq)$ is a partial order that is not total, but has a minimum element $\varnothing$ and a maximum element $\mathcal{N}$.
- $(A^*, \preceq)$ is a partial order that is not total, but has a minimum element ε.

Linearising finite partial orders

Proposition

For any finite partial order $(X, \leq)$*, there is a total order* $\leq'$ *on* X *that extends* $\leq$*, i.e.* $x \leq y \implies x \leq' y$.

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Proof.

By induction on $|X|$, the size of X.

- For the base case, when X is empty, $\leq$ is trivially already total.
- For the inductive step, let $|X| = n + 1$.
  Since X is finite, let $x \in X$ be a minimal element, *i.e.* if $y \leq x$ then $y = x$.
- By the inductive hypothesis, let $\leq'$ be a total order on $X \setminus \{x\}$ extending $\leq$.
- Now construct the total order $\leq''$ on X extending $\leq'$ by setting:

$$x \leq'' y \text{ for all } y \in X.$$

□

Extending to the infinite case

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Let $(X, \leq)$ be a partial order, and write propositional variables p_{xy} for $x, y \in X$. Define the set Γ to consist of the following formulae:

1. p_{xy} for every $x, y \in X$ s.t. $x \leq y$.
2. p_{xx} for every $x \in X$.
3. $\neg p_{xy} \vee \neg p_{yx}$ for every $x, y \in X$ with $x \neq y$.
4. $p_{xy} \to p_{yz} \to p_{xz}$ for every $x, y, z \in X$.
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Let $\Gamma' \subseteq \Gamma$ be finite. Notice that $\Gamma' \setminus \{\text{axioms of the form (5)} : p_{xy} \vee p_{yx}\}$ is already satisfied by some finite fragment of $(X, \leq)$.

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Applying the finite case of the theorem (*i.e.* the Proposition on the previous slide) to this fragment induces an assignment that satisfies all of Γ'.

Hence Γ is finitely satisfiable so, by compactness, it is actually satisfiable, say by an assignment α. This, in turn, induces a total order on X extending $\leq$ □

Some powerful corollaries

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The compactness theorem has numerous fascinating consequences in combinatorics and algebra:

Corollary (of Compactness)

- *$\mathcal{P}(\mathcal{N})$ can be totally ordered in a manner consistent with $\subseteq$.*
- *The finite Ramsey theorem is a consequence of the infinite Ramsey theorem.*
- *Assuming the finite four colour theorem, every infinite map can be coloured with only four colours.*
- *König's lemma: every infinite finitely branching tree has an infinite path.*

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Exercise: if, like me, you think this is cool, try proving these yourself!

Outline

The **language of arithmetic** is $\mathcal{L}_A := \{0, s, +, \cdot\}$. The 'standard structure' is $\mathcal{N} = (\mathbb{N}, 0_\mathcal{N} := 0 \in \mathbb{N}, s_\mathcal{N} : n \mapsto n{+}1, +_\mathcal{N} : (m, n) \mapsto m{+}n, \cdot_\mathcal{N} : (m, n) \mapsto m{\times}n)$.

Definition (Peano arithmetic (PA))

PA is a theory over $\mathcal{L}_A$ containing the axioms:

- $\forall x \neg(0 = s(x))$
- $\forall x, y(s(x) = s(y) \to x = y)$
- $\forall x(x + 0 = x)$
- $\forall x, y(x + s(y) = s(x + y))$
- $\forall x(x \cdot 0 = 0)$
- $\forall x, y(x \cdot s(y) = x \cdot y + x)$

PA also has an axiom schema of **induction**: for each formula A with $\mathsf{FV}(A) = \{x, y_1, \dots, y_k\}$ there is an axiom:

$$\forall y_1, \dots, y_k \, (A[0/x] \wedge \forall x(A \to A[s(x)/x]) \to \forall x.A)$$

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The axioms of PA* contain all the axioms of PA, the axiom $x < y \leftrightarrow \exists z\, x + s(z) = y$, and all axioms of the form,

$$\omega > \bar{n}$$

where $\bar{n} :\equiv s^{(n)}(0)$ for $n \in \mathcal{N}$.

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where $\bar{n} :\equiv s^{(n)}(0)$ for $n \in \mathcal{N}$.

Any *finite* subset $\Gamma \subseteq$ PA* is satisfied by $\mathcal{N}$. We only need to take care of the constant ω, which is interpreted by any $N \in \mathbb{N}$ larger than all $n_1, \ldots, n_k$ where

$$\omega > \bar{n}_1, \ldots, \omega > \bar{n}_k$$

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are the only such axioms in Γ.

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are the only such axioms in Γ.

By the compactness theorem, we must have $\mathcal{N}^* \models$ PA* for some structure $\mathcal{N}^*$, and hence also $\mathcal{N}^* \models$ PA.

NB: in $\mathcal{N}^*$, the interpretation of ω must dominate all numerals $\bar{n}$!

0 1 2 . . . . . ω $\omega+1$ $\omega+2$

In the first-order setting, we have to accept the existence of nonstandard models $\mathcal{N}^*$. (cf. the *Lowenheim-Skolem theorem*, see [Bays, 2014])

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This problem can be avoided by formulating of Peano's postulates using **second-order logic**, which in addition to talking about things $x, y, z, \ldots$ allows us reason about sets of things $X, Y, Z, \ldots$.

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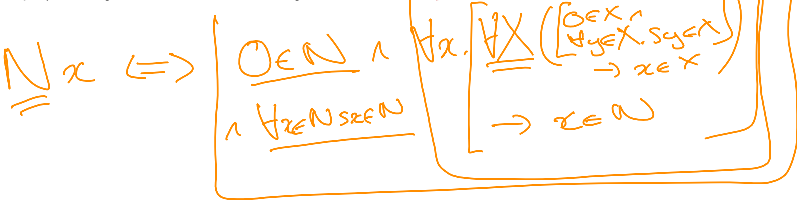
This problem can be avoided by formulating of Peano's postulates using **second-order logic**, which in addition to talking about things $x, y, z, \ldots$ allows us reason about sets of things $X, Y, Z, \ldots$.

The price to pay is high: second-order logic has no complete axiomatisation.



Outline

1. Show that the following forms of consistency are equivalent:

$$
\begin{array}{lcl}
(1) & : & \Gamma \nvdash \bot \\
(2) & : & \text{for no formula } A,\ \Gamma \vdash A \text{ and } \Gamma \vdash \neg A \\
(3) & : & \text{there is a formula } A \text{ s.t. } \Gamma \nvdash A
\end{array}
$$

2. Show that the following formulas are equivalent,

$$((A \to B) \to C) \to D$$
$$(A \to p) \to (p \to A) \to ((p \to B) \to C) \to D$$

   where p does not occur in any of A, B, C, D.

   Hint: Under soundness and completeness, you may use either syntactic or semantic means, or a combination!

3. (**Hard**). A **graph** is a structure $G = (V, E)$ where $E \subseteq V \times V$. For $k \in \mathcal{N}$ we say that G is k-**colourable** if there is a 'colouring' function $c : V \to \{1, \ldots, k\}$ such that, whenever $(u, v) \in E$ we have $c(u) \neq c(v)$.

   Show, using the compactness theorem, that a (possibly infinite) graph is k-colourable if and only if every finite subgraph of it is k-colourable.

Outline

References I

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