Introduction to Proof Theory

Lecture 3 - Gentzen's sequent calculus

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These slides are available at `http://www.anupamdas.com/mgs21`.

Based on slides from ESSLLI'18, prepared with Thomas Powell.

Outline

Proofs, provability and systems

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But there are many other ones to consider!

- **Natural deduction.** (Normalisation as computation)
- **Sequent calculus.** (General metalogical framework)
- **Resolution.** (Automated theorem proving)
- **Analytic tableaux.** (Semantics)
- Systems with **Extension** or **Substitution**. (Proof complexity)
- **Algebraic** and **Geometric** systems. (Structural proof theory)
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Question: Why is one system better than another? We will consider just one desideratum here...

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This is a serious issue, but reflects the difficulty of the problem at hand:

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DESIDERATUM: Proofs in $\mathcal{F}$ are rather short (believe it or not): can we trade off proof size for less nondeterminism in proof search?

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Question

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   - Any domino must cover a white square and a black square.
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NB: If you want more of these, come speak to me!

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BRINGING DUALITY TO THE FOREFRONT

Duality is at the heart of proof theory; we should combine this with deduction.

Recall the following notation:

$\Gamma \vdash A$: "all the formulas of Γ imply A"

Gentzen's insight was to dualise this notion:

$\Gamma \vdash \Delta$: "all the formulae of Γ imply some of the formulae of Δ"

Sequents

The desire of “fully dualised” systems motivates the following notion:

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set with multiple occurrences

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We write sequents as $A_1, \ldots, A_m \vdash B_1, \ldots, B_n$, and interpret them as:

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When Γ is empty, we identify it with $\top$. When Δ is empty we identify it with $\bot$. So:

- $\Gamma \vdash$ means "Γ is inconsistent".
- $\vdash \Delta$ means "some formula in Δ is true".

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Gentzen proposed a system called LK, which notably enjoys the following completeness property:

Theorem (Subformula property, Gentzen '34)

If A is valid, it has an LK *proof containing only subformulae of A (up to substitution of terms for free variables).*

Outline

Initial and structural rules

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$$\text{id}\,\frac{}{A \vdash A} \qquad \textit{w-l}\,\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \textit{w-r}\,\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \textit{c-l}\,\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \textit{c-r}\,\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Initial and structural rules

"$\Gamma \vdash \Delta$" : "$\bigwedge \Gamma \to \bigvee \Delta$"
$\Rightarrow \bigwedge \Gamma \to \bigvee \Delta \vee A$

$\Gamma, A := \Gamma \cup \{A\}$

$$\text{id}\ \frac{}{A \vdash A} \qquad w\text{-}l\ \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad w\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\ \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

$A \vee A \Downarrow A$

- (**Identity**, id) corresponds to the axiom $A \to A$ we derived in Lecture 1.
- (**Weakening**, w) corresponds to a relaxation of any deduction:

$$\textit{if } \Gamma \vdash B \textit{ then also } \Gamma, A \vdash B$$

The right version is just the dual version.

$\Gamma \vdash B$
$\Gamma, A \vdash B$

$\Gamma \quad \neg A$

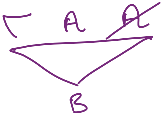
- (**Contraction**, c) corresponds to the tautologies:

$$A \to (A \wedge A) \qquad \text{and} \qquad (A \vee A) \to A$$

Alternatively: if I already have A as a hypothesis, I do not need two copies of it!

w-l: $A \wedge B \to A$

w-r: $A \to A \vee B$



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Propositional connectives

We only present the rules for $\rightarrow$ and $\bot$, since that was our basis for $\mathcal{F}$.

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There is only one rule for $\bot$:

$$\bot\text{-}l \; \frac{}{\bot \vdash} \qquad \left(\textit{optional} : \; \bot\text{-}r \; \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \right)$$

This is *ex falso quod libet*, that we proved before: $\bot \rightarrow A$ is valid for any formula A.
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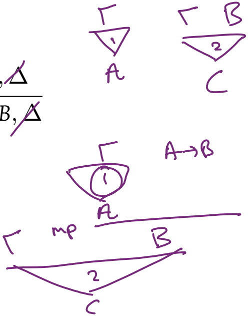
We have two symmetric rules for implication:

$$\to\text{-}l\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$



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- The *left* rule allows us to 'compose' derivations.
- The *right* rule is just a generalisation of the deduction theorem.

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Quantifier rules

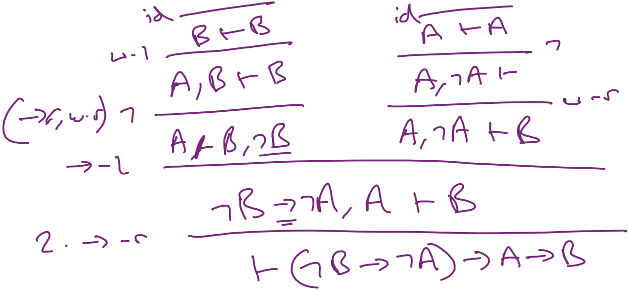
Importantly, LK has the following rules for the quantifiers:

$$\forall\text{-}l\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A}\ y \notin \mathsf{FV}(\Gamma, \Delta, A)$$

$$\exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}\ y \notin \mathsf{FV}(\Gamma, \Delta, A)$$

- In $\exists$-l and $\forall$-r the variable x may not occur free in Γ, Δ.
- Notice the duality between $\exists$ and $\forall$ here.

Outline


id
B ⊢ B
w-l
A, B ⊢ B
(→r, w-r) ¬
A ⊢ B, ¬B
id
A ⊢ A
¬
A, ¬A ⊢
w-r
A, ¬A ⊢ B
→-l
¬B → ¬A, A ⊢ B
2. →-r
⊢ (¬B → ¬A) → A → B

Contraposition

$$
{\scriptstyle \to\text{-}r}\;\dfrac{
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{\scriptstyle \to\text{-}l}\;\dfrac{
{\scriptstyle \to\text{-}r}\;\dfrac{{\scriptstyle w}\;\dfrac{{\scriptstyle \text{id}}\;\dfrac{}{B \vdash B}}{A, B \vdash B}}{A \vdash B, \neg B}
\qquad
{\scriptstyle \to\text{-}l}\;\dfrac{{\scriptstyle w}\;\dfrac{{\scriptstyle \text{id}}\;\dfrac{}{A \vdash A}}{A \vdash A, B} \quad {\scriptstyle w}\;\dfrac{{\scriptstyle \bot\text{-}l}\;\dfrac{}{\bot \vdash}}{\bot \vdash B}}{A, \neg A \vdash B}
}{\neg B \to \neg A, A \vdash B}
}{\neg B \to \neg A \vdash A \to B}
}{\vdash (\neg B \to \neg A) \to A \to B}
$$

Pierce's law

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A, B}\ w}{\vdash A, A \to B} \qquad \overline{A \vdash A}}{(A \to B) \to A \vdash A}\ \to l$$

$$\dfrac{(A \to B) \to A \vdash A}{\vdash ((A \to B) \to A) \to A}\ \to r$$

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Drinker's paradox

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D(v) \vdash D(v)}\,\textsf{id}}{D(u), D(v) \vdash D(v), \forall y.D(y)}\,w}{D(u) \vdash D(v), D(v) \to \forall y.D(y)}\,{\to}\text{-}r}{D(u) \vdash D(v), \exists x.(D(x) \to \forall y.D(y))}\,\exists\text{-}r}{D(u) \vdash \forall x.D(x), \exists x.(D(x) \to \forall y.D(y))}\,\forall\text{-}r}{\vdash D(u) \to \forall y.D(y), \exists x.(D(x) \to \forall y.D(y))}\,{\to}\text{-}r}{\vdash \exists x.(D(x) \to \forall y.D(y)), \exists x.(D(x) \to \forall y.D(y))}\,\exists\text{-}r}{\vdash \exists x.(D(x) \to \forall y.D(y))}\,c
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There is one crucial exception:

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(This concludes the definition of LK.)

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Notice that A here, the cut-formula, had to be guessed.

This is very similar to the situation for *modus ponens* and, indeed, allows us to easily simulate $\mathcal{F}$ proofs, or prove completeness at all…

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NB: (1) may also be proved by simulation, but a little harder!

The *Hauptsatz*

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Perspectives

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Its applications are numerous:

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- Automating proof search. (logic programming)
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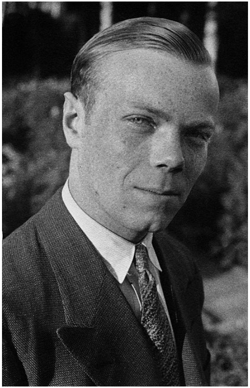
As my old supervisor used to say: "*there is no such thing as a free lunch*".

Theorem (Statman '79, Orevkov '82)

Cut-elimination necessarily has a non-elementary cost in proof size.

Herbrand's theorem



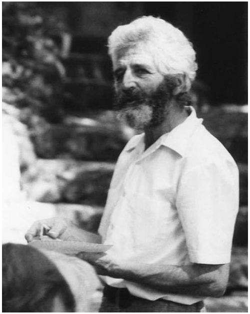
Theorem (Herbrand)

If $\models \exists x.A(x)$, *for A quantifier-free, then there is a finite set* $\{t_i\}_{i=1}^{n}$ *s.t.* $\models A(t_1) \vee \cdots \vee A(t_n)$.

Simple example: even numbers exist (blank slide)

Informal example: $\exists a, b \notin \mathbb{Q}. a^b \in \mathbb{Q}$ (blank slide)

Craig interpolation



Theorem (Craig Interpolation)

If $\vDash A \rightarrow B$*, there is some* I *in the common language of* A *and* B *s.t.* $\vDash A \rightarrow I$ *and* $\vDash I \rightarrow B$*.*

Example: disjoint **NP**-pairs (blank slide)

Outline

A remarkable observation

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Corollary

We have the following immediate consequences:

- *Intuitionistic propositional logic is decidable (even* **PSPACE***-complete).*
- *Intuitionistic logic enjoys the disjunction and existential properties:*

$$\begin{array}{lcl} \vdash_{\mathsf{LJ}} A \vee B & \Longrightarrow & \vdash_{\mathsf{LJ}} A \text{ or } \vdash_{\mathsf{LJ}} B \\ \vdash_{\mathsf{LJ}} \exists x A & \Longrightarrow & \vdash_{\mathsf{LJ}} A[t/x] \text{ for some term } t \end{array}$$

- *Craig interpolation still holds for intuitionistic logic.*

Example: Peirce's law is not intuitionistically valid (blank slide)

Outline

The system LK

INITIAL AND STRUCTURAL RULES

$$\text{id}\,\frac{}{A \vdash A} \qquad w\text{-}l\,\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad w\text{-}r\,\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\,\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad c\text{-}r\,\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

PROPOSITIONAL RULES:

$$\bot\text{-}l\,\frac{}{\bot \vdash} \qquad \left(\bot\text{-}r\,\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot}\right) \qquad \to\text{-}l\,\frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\,\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

QUANTIFIER RULES:

$$\forall\text{-}l\,\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\,\frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\,\frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\,\frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

CUT RULE:

$$cut\,\frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

1. We saw that modus ponens could be simulated in LK. Give cut-free LK proofs for each of the three propositional axioms of $\mathcal{F}$:

$$\begin{array}{rl} (wk) & A \to (B \to A) \\ (dist) & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\ (neg) & ((A \to \bot) \to \bot) \to A \end{array}$$

2. What about the quantifier axioms and rule:

$$gen\ \frac{A}{\forall x A} \qquad \begin{array}{l} \forall x.A \to A[t/x] \\ \forall x.(A \to B) \to (A \to \forall x.B) \quad \text{as long as } x \notin \mathsf{FV}(A) \end{array}$$

   (**NB:** this concludes the proof of completeness of LK).

3. What possible rules might we need if we had $\vee$ and $\wedge$ in our language?

4. (**Puzzle**). There are 100 ants on a 1m stick, facing either end of the stick. They start moving at 1m/s and, every time they collide, they perfectly rebound. How long does it take for all the ants to fall off the stick (in the worst case)?

5. (**Still hard**). Revisiting Exercise 4 (Propositional logic) from Lecture 1: can you now show that Peirce's law, $((A \to B) \to A) \to A$, is not intuitionistically valid?

   Hint: use completeness of cut-free proof search in LJ for intuitionistic logic.

Outline

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