

INTRODUCTION TO PROOF THEORY
Lecture 3 - Gentzen's sequent calculus

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These slides are available at <http://www.anupamdas.com/mgs21>.

Based on slides from ESSLLI'18, prepared with Thomas Powell.

- 1 Proof systems and analyticity
- 2 Extending deduction: a structural approach
- 3 The system (cut-free) LK
- 4 Examples of cut-free proofs: almost automatic proof search
- 5 Completeness, the cut, and its elimination
- 6 Consequences
- 7 Recovering a calculus for intuitionistic logic
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But there are **many other ones** to consider!

- **Natural deduction.** (Normalisation as computation)
- **Sequent calculus.** (General metalogical framework)
- **Resolution.** (Automated theorem proving)
- **Analytic tableaux.** (Semantics)
- Systems with **Extension** or **Substitution**. (Proof complexity)
- **Algebraic** and **Geometric** systems. (Structural proof theory)
- ...

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QUESTION: Why is one system **better** than another? We will consider just one desideratum here...

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This is a serious issue, but reflects the difficulty of the problem at hand:

Proposition

- *Checking whether a propositional formula is valid is **coNP-complete**.*
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DESIDERATUM: Proofs in \mathcal{F} are rather **short** (believe it or not): can we **trade off** proof size for **less nondeterminism** in proof search?

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Question

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 - Any domino must cover a **white** square and a **black** square.
 - Thus any tiling covers the **same number** of black and white squares.
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NB: If you want more of these, come speak to me!

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Gentzen's insight was to **dualise** this notion:

$\Gamma \vdash \Delta$: “all the formulae of Γ imply some of the formulae of Δ ”

The desire of “fully dualised” systems motivates the following notion:

Definition (Sequents)

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We write sequents as $A_1, \dots, A_m \vdash B_1, \dots, B_n$, and **interpret** them as:

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When Γ is empty, we **identify** it with \top . When Δ is empty we **identify** it with \perp . So:

- $\Gamma \vdash$ means “ Γ is inconsistent”.
- $\vdash \Delta$ means “some formula in Δ is true”.

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A calculus for metatheory

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Gentzen proposed a system called LK, which notably enjoys the following **completeness** property:

Theorem (Subformula property, Gentzen '34)

*If A is valid, it has an LK proof containing **only subformulae** of A (up to substitution of terms for free variables).*

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Initial and structural rules

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$$\text{id} \frac{}{A \vdash A}$$

$$w-l \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

$$w-r \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}$$

$$c-l \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

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- (**Identity**, id) corresponds to the **axiom** $A \rightarrow A$ we derived in Lecture 1.
- (**Weakening**, w) corresponds to a **relaxation** of any deduction:

if $\Gamma \vdash B$ then also $\Gamma, A \vdash B$

The right version is just the **dual** version.

- (**Contraction**, c) corresponds to the tautologies:

$$A \rightarrow (A \wedge A) \quad \text{and} \quad (A \vee A) \rightarrow A$$

Alternatively: if I already have A as a hypothesis, I do not need **two copies** of it!

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Propositional connectives

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There is only one rule for \perp :

$$\perp\text{-}l \frac{}{\perp \vdash} \quad \left(\text{optional : } \perp\text{-}r \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \perp} \right)$$

This is *ex falso quod libet*, that we proved before: $\perp \rightarrow A$ is valid for any formula A .
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We have two **symmetric** rules for implication:

$$\rightarrow\text{-}l \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \quad \rightarrow\text{-}r \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}$$

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- The *left* rule allows us to ‘**compose**’ derivations.
- The *right* rule is just a generalisation of the **deduction theorem**.

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Quantifier rules

Importantly, LK has the following rules for the **quantifiers**:

$$\begin{array}{l} \forall\text{-}l \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \quad \forall\text{-}r \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \quad y \notin \text{FV}(\Gamma, \Delta, A) \\ \exists\text{-}r \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \quad \exists\text{-}l \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta} \quad y \notin \text{FV}(\Gamma, \Delta, A) \end{array}$$

- In $\exists\text{-}l$ and $\forall\text{-}r$ the variable x may not occur free in Γ, Δ .
- Notice the **duality** between \exists and \forall here.

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$$\begin{array}{c} \text{id} \frac{}{B \vdash B} \quad \text{id} \frac{}{A \vdash A} \quad \perp\text{-l} \frac{}{\perp \vdash} \\ \text{w} \frac{}{A, B \vdash B} \quad \text{w} \frac{}{A \vdash A, B} \quad \text{w} \frac{}{\perp \vdash B} \\ \rightarrow\text{-r} \frac{}{A \vdash B, \neg B} \quad \rightarrow\text{-l} \frac{}{A, \neg A \vdash B} \\ \rightarrow\text{-l} \frac{}{\neg B \rightarrow \neg A, A \vdash B} \\ \rightarrow\text{-r} \frac{}{\neg B \rightarrow \neg A \vdash A \rightarrow B} \\ \rightarrow\text{-r} \frac{}{\vdash (\neg B \rightarrow \neg A) \rightarrow A \rightarrow B} \end{array}$$

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 \end{array}$$

Drinker's paradox

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$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\text{id}}{D(v) \vdash D(v)}}{D(u), D(v) \vdash D(v), \forall y.D(y)}}{\rightarrow-r}{D(u) \vdash D(v), D(v) \rightarrow \forall y.D(y)}}{\exists-r}{D(u) \vdash D(v), \exists x.(D(x) \rightarrow \forall y.D(y))}}{\forall-r}{D(u) \vdash \forall x.D(x), \exists x.(D(x) \rightarrow \forall y.D(y))}}{\rightarrow-r}{\vdash D(u) \rightarrow \forall y.D(y), \exists x.(D(x) \rightarrow \forall y.D(y))}}{\exists-r}{\vdash \exists x.(D(x) \rightarrow \forall y.D(y)), \exists x.(D(x) \rightarrow \forall y.D(y))}}{c}{\vdash \exists x.(D(x) \rightarrow \forall y.D(y))}$$

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There is one crucial **exception**:

$$\text{cut} \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

(This concludes the definition of LK.)

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(This concludes the definition of LK.)

Notice that A here, the **cut-formula**, had to be **guessed**.

This is very similar to the situation for *modus ponens* and, indeed, allows us to easily **simulate** \mathcal{F} proofs, or prove **completeness** at all...

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□

NB: (1) may also be proved by simulation, but a little harder!



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Corollary (Subformula property)

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Its applications are numerous:

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- **Automating** proof search. (logic programming)
- Cut-elimination as a **computational process**. (*Curry-Howard* correspondence)
- **Metalogical** results. (structural proof theory)

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As my old supervisor used to say: “*there is no such thing as a free lunch*”.

Theorem (Statman '79, Orevkov '82)

*Cut-elimination necessarily has a **non-elementary cost** in proof size.*

Herbrand's theorem

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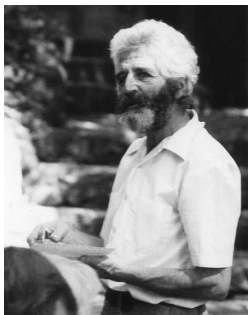
Theorem (Herbrand)

If $\models \exists x.A(x)$, for A quantifier-free, then there is a finite set $\{t_i\}_{i=1}^n$ s.t. $\models A(t_1) \vee \dots \vee A(t_n)$.

Simple example: even numbers exist (blank slide)

Informal example: $\exists a, b \notin \mathbb{Q}. a^b \in \mathbb{Q}$ (blank slide)

Craig interpolation



Theorem (Craig Interpolation)

If $\models A \rightarrow B$, there is some I in the *common language* of A and B s.t. $\models A \rightarrow I$ and $\models I \rightarrow B$.

Example: disjoint **NP**-pairs (blank slide)

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A remarkable observation

Gentzen noticed a startling fact:

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Let LJ be the restriction of LK where all sequents have exactly *one formula* on the RHS.

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LJ is *sound and complete* for intuitionistic logic.

Moreover, *cut-elimination still holds!*

Corollary

We have the following immediate consequences:

- Intuitionistic propositional logic is *decidable* (even **PSPACE**-complete).
- Intuitionistic logic enjoys the *disjunction and existential properties*:

$$\begin{aligned} \vdash_{\text{LJ}} A \vee B &\implies \vdash_{\text{LJ}} A \text{ or } \vdash_{\text{LJ}} B \\ \vdash_{\text{LJ}} \exists x A &\implies \vdash_{\text{LJ}} A[t/x] \text{ for some term } t \end{aligned}$$

- *Craig interpolation still holds for intuitionistic logic.*

Example: Peirce's law is not intuitionistically valid (blank slide)

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The system LK

INITIAL AND STRUCTURAL RULES

$$\text{id} \frac{}{A \vdash A} \quad \text{w-l} \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \text{w-r} \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \quad \text{c-l} \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \quad \text{c-r} \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

PROPOSITIONAL RULES:

$$\perp\text{-l} \frac{}{\perp \vdash} \quad \left(\perp\text{-r} \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \perp} \right) \quad \rightarrow\text{-l} \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta} \quad \rightarrow\text{-r} \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}$$

QUANTIFIER RULES:

$$\forall\text{-l} \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \quad \forall\text{-r} \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \quad \exists\text{-r} \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \quad \exists\text{-l} \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \text{FV}(\Gamma, \Delta, A)$.

CUT RULE:

$$\text{cut} \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Exercises, Lecture 3

- ① We saw that modus ponens could be simulated in LK. Give cut-free LK proofs for each of the three propositional axioms of \mathcal{F} :

$$\begin{aligned}(\text{wk}) \quad & A \rightarrow (B \rightarrow A) \\(\text{dist}) \quad & (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \\(\text{neg}) \quad & ((A \rightarrow \perp) \rightarrow \perp) \rightarrow A\end{aligned}$$

- ② What about the quantifier axioms and rule:

$$\text{gen} \frac{A}{\forall x A} \quad \forall x. A \rightarrow A[t/x] \\ \forall x. (A \rightarrow B) \rightarrow (A \rightarrow \forall x. B) \quad \text{as long as } x \notin \text{FV}(A)$$

(NB: this concludes the proof of completeness of LK).

- ③ What possible rules might we need if we had \vee and \wedge in our language?
- ④ (**Puzzle**). There are 100 ants on a 1m stick, facing either end of the stick. They start moving at 1m/s and, every time they collide, they perfectly rebound. How long does it take for all the ants to fall off the stick (in the worst case)?
- ⑤ (**Still hard**). Revisiting Exercise 4 (Propositional logic) from Lecture 1: can you now show that Peirce's law, $((A \rightarrow B) \rightarrow A) \rightarrow A$, is not intuitionistically valid?

Hint: use completeness of cut-free proof search in LJ for intuitionistic logic.

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