Introduction to Proof Theory

Lecture 4 - *Hauptsatz*: the cut-elimination theorem

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These slides are available at `http://www.anupamdas.com/mgs21`.

Based on slides from ESSLLI'18, prepared with Thomas Powell.

Recap: the system LK

Initial and structural rules

$$\text{id}\ \frac{}{A \vdash A} \qquad \textit{w-l}\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \textit{w-r}\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \textit{c-l}\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad \textit{c-r}\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

Propositional rules:

$$\bot\text{-}l\ \frac{}{\bot \vdash} \qquad \bot\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \qquad \to\text{-}l\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

Quantifier rules:

$$\forall\text{-}l\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

Cut rule:

$$\textit{cut}\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Outline

1. The problem of cut-elimination
2. Key and commutative logical reductions
3. Structural rules: a bit of a headache
4. A strategy and an induction measure
5. Proofs of some consequences
6. References

The subject matter of *structural proof theory*

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Structural proof theory *is the study of normal forms and normalisation*

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Not only does it yield our much sought after analytic system, it is one of the most powerful tools in all of logic

"*cut-elimination is the most beautiful result with the most horrible proof*"

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"*cut-elimination is the most beautiful result with the most horrible proof*"

This is one of the hardest termination arguments in graduate mathematics and computer science!

Theorem (*Hauptsatz*, Gentzen '34)

Every theorem of LK *has a proof that does not use the cut rule.*

Corollary (Subformula property)

Every theorem of LK *has a proof that contains only subformulae of it, up to substitution of variables for terms.*

How cut-elimination works

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Basic idea, in a nutshell

*"push" the cuts **upwards** in a proof, until they disappear completely!*

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I will outline the argument, but understanding comes through practice! I recommend standard references for detailed proofs:

- [Buss, 1998]. *Handbook of Proof Theory*.
- [Troelstra and Schwichtenberg, 1996]. *Basic Proof Theory*.
- [Negri and von Plato, 2001]. *Structural Proof Theory*.
- [Szabo, 1972]. *The Collected Papers of Gerhard Gentzen*.
- Or do it *yourself*!

Outline

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Key propositional case

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$$
\mathit{cut}\;\dfrac{
{\scriptstyle \to\text{-}r}\;\dfrac{\overset{\overset{1}{\triangledown}}{\Gamma, A \vdash \Delta, B}}{\Gamma \vdash \Delta, \boxed{A \to B}}
\qquad
{\scriptstyle \to\text{-}l}\;\dfrac{\overset{\overset{2}{\triangledown}}{\Gamma' \vdash \Delta', A} \qquad \overset{\overset{3}{\triangledown}}{\Gamma', B \vdash \Delta'}}{\Gamma', \boxed{A \to B} \vdash \Delta'}
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

$$\dfrac{\dfrac{\overset{1}{\Gamma, A \vdash \Delta, B}}{\Gamma \vdash \Delta, A \to B}\;{\to}\text{-}r \qquad \dfrac{\overset{2}{\Gamma' \vdash \Delta', A} \quad \overset{3}{\Gamma', B \vdash \Delta'}}{\Gamma', A \to B \vdash \Delta'}\;{\to}\text{-}l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;\mathit{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\overset{2}{\Gamma' \vdash \Delta', A} \quad \overset{1}{\Gamma, A \vdash \Delta, B}}{\Gamma, \Gamma' \vdash \Delta, \Delta', B}\;\mathit{cut} \qquad \overset{3}{\Gamma', B \vdash \Delta'}}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}\;\mathit{cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;c$$

$A \to B$

$\underline{\underline{A}}$ $\quad$ $\underline{\underline{B}}$

Key quantifier case

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$$
\textit{cut}\;\dfrac{\exists\text{-}r\;\dfrac{\overset{1}{\Gamma \vdash \Delta, A(t)}}{\Gamma \vdash \Delta, \exists x.A(x)} \qquad \exists\text{-}l\;\dfrac{\overset{2(y)}{\Gamma', A(x) \vdash \Delta'}}{\Gamma', \exists x.A(x) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

Key quantifier case

$$\dfrac{\dfrac{\begin{matrix}1\\ \vdots \\ \Gamma \vdash \Delta, \boxed{A(t)}\end{matrix}}{\Gamma \vdash \Delta, \underline{\exists x.A(x)}}\,\exists\text{-}r \qquad \dfrac{\begin{matrix}2(y)\\ \vdots \\ \Gamma', \underline{A(y)} \vdash \Delta'\end{matrix}}{\Gamma', \underline{\exists x.A(x)} \vdash \Delta'}\,\exists\text{-}l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut \quad \rightsquigarrow \quad \dfrac{\begin{matrix}1\\ \vdots \\ \Gamma \vdash \Delta, A(t)\end{matrix} \qquad \begin{matrix}2(t)\\ \vdots \\ \Gamma', A(t) \vdash \Delta'\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut$$

$$\exists x\, A(x) \rightsquigarrow A(t)$$

$$\text{cut}\ \dfrac{\exists\text{-}r\ \dfrac{\overset{\textstyle 1}{\nabla}\atop \Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x.A(x)} \qquad \exists\text{-}l\ \dfrac{\overset{\textstyle 2(y)}{\nabla}\atop \Gamma', A(x) \vdash \Delta'}{\Gamma', \exists x.A(x) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad \rightsquigarrow \quad \text{cut}\ \dfrac{\overset{\textstyle 1}{\nabla}\atop \Gamma \vdash \Delta, A(t) \qquad \overset{\textstyle 2(t)}{\nabla}\atop \Gamma', A(t) \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

- Implicitly assume free variable conditions are respected.
- Write $2(t)$ for the derivation arising from $2(y)$ by replacing every free occurrence of y by t.

$\underline{\exists x\, A(x)} \rightsquigarrow \underline{A(t)}$

Key quantifier case

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- Implicitly assume free variable conditions are respected.
- Write $2(t)$ for the derivation arising from $2(y)$ by replacing every free occurrence of y by t.

NB: The key $\forall$ case is dual.

(Example of) commutative cases

$$
\dfrac{\dfrac{1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\dfrac{2}{\Gamma', A, B \vdash \Delta', C}}{\Gamma', A \vdash \Delta', B \to C}\ \to\text{-}r}{\Gamma, \Gamma' \vdash \Delta, \Delta', B \to C}\ cut
$$

→l commutative:

→l

cut

$$
\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{1}{\Gamma \vdash \Delta, A} \qquad \dfrac{2}{\Gamma', A, B \vdash \Delta', C}}{\Gamma, \Gamma', B \vdash \Delta, \Delta', C}\ cut}{\Gamma, \Gamma' \vdash \Delta, \Delta', B \to C}\ \to r
$$

Outline

Weakening: enter non-confluence

$$\dfrac{\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A}\,\textit{w-r} \qquad \overset{\nabla_2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\textit{cut} \quad \rightsquigarrow \quad \boxed{\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w}$$

– no cut

$$\dfrac{\boxed{\overset{\nabla_1}{\Gamma \vdash \Delta, A}} \qquad \boxed{\dfrac{\overset{\nabla_2}{\Gamma' \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\,\textit{w-l}}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\textit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\nabla_2}{\Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w$$

Weakening: enter non-confluence II

NB: cut-reduction is not confluent!

$$\boxed{\dfrac{\dfrac{\triangledown_1}{\Gamma \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, w} \quad \leftsquigarrow \quad \dfrac{\dfrac{\dfrac{\triangledown_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A}\, \textit{w-r} \qquad \dfrac{\dfrac{\triangledown_2}{\Gamma' \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\, \textit{w-l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, \textit{cut} \quad \rightsquigarrow \quad \boxed{\dfrac{\dfrac{\triangledown_2}{\Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\, w}$$

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$$
w\,\dfrac{\begin{array}{c}\overset{1}{\triangledown}\\ \Gamma \vdash \Delta\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
\quad\rightsquigarrow\!\!\!\!\leftarrow\quad
cut\,\dfrac{w\text{-}r\,\dfrac{\begin{array}{c}\overset{1}{\triangledown}\\ \Gamma \vdash \Delta\end{array}}{\Gamma \vdash \Delta, A} \qquad w\text{-}l\,\dfrac{\begin{array}{c}\overset{2}{\triangledown}\\ \Gamma' \vdash \Delta'\end{array}}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
\quad\rightsquigarrow\quad
w\,\dfrac{\begin{array}{c}\overset{2}{\triangledown}\\ \Gamma' \vdash \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

This is known as **Lafont's counterexample**.

Contraction: enter non-termination

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One particularly nasty issue is:

$$
\mathit{cut}\;\dfrac{\overset{1}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\;\dfrac{\overset{2}{\Gamma', A, A \vdash \Delta'}}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
\quad\rightsquigarrow\quad
c\;\dfrac{\mathit{cut}\;\dfrac{\overset{1}{\Gamma \vdash \Delta, A} \qquad \mathit{cut}\;\dfrac{\overset{1}{\Gamma \vdash \Delta, A} \qquad \overset{2}{\Gamma', A, A \vdash \Delta'}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

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Here the entire subproof 1 is duplicated!

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$$
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\quad\rightsquigarrow\quad
{\scriptstyle c}\,\dfrac{{\scriptstyle cut}\,\dfrac{\dfrac{\nabla_1}{\Gamma \vdash \Delta, A} \qquad {\scriptstyle cut}\,\dfrac{\dfrac{\nabla_1}{\Gamma \vdash \Delta, A} \quad \dfrac{\nabla_2}{\Gamma', A, A \vdash \Delta'}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

Here the entire subproof 1 is duplicated!

NB: the reduction for *c-r* is dual.

Contraction: enter non-termination II

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The sequent in red is trivial.

Even worse, just blindly applying permutation rules will not work:

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The sequent in red is trivial.

NB: cut-reduction is not strongly normalising.

terminating

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Outline

The critical lemma via left-then-right strategy

Lemma
For any proof

$$\dfrac{\dfrac{\pi}{\Gamma \vdash \Delta, A} \quad \dfrac{\pi'}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \textit{cut} \tag{1}$$

where the subproofs π and π' have only cuts on formulas smaller than A, there is a proof of $\Gamma, \Gamma' \vdash \Delta, \Delta'$ with only cuts on formulas smaller than A.

key

smaller cut

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|cut formula| < |A|

Proof idea.

- First push the cut up π' maximally (by induction on $|\pi'|$).
- For any topmost resulting cut, push up the LHS maximally (by induction on |LHS|) and apply key reduction to leave a smaller cut.
- Repeat the process for remaining cuts. □

The critical lemma via left-then-right strategy

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$$\dfrac{\overset{\pi}{\Gamma \vdash \Delta, A} \quad \overset{\pi'}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathit{cut} \tag{1}$$

where the subproofs π and π' have only cuts on formulas smaller than A, there is a proof of $\Gamma, \Gamma' \vdash \Delta, \Delta'$ *with only cuts on formulas smaller than A.*

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- First push the cut up π' maximally (by induction on $|\pi'|$).
- For any topmost resulting cut, push up the LHS maximally (by induction on |LHS|) and apply key reduction to leave a smaller cut.
- Repeat the process for remaining cuts. □

More details: ...will be given in the exercise session.

An induction measure

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sets with multiple occurrences
lists quotiented by order

Definition (Multiset measure)

For two multisets M, N of natural numbers, we say $M \prec N$ if I may transform N into M by the continually applying the following operation:

$$\textit{delete a number } n \in N \textit{ and insert any number of numbers} < n$$

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Example

$\{1, 2, 3, 4, 4, 5\} \prec \{2, 5, 5\}$ and $\{1, 1, 2, 2, 2, 2\} \prec \{1, 2, 3\}$.

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NB: This can also be seen as a lexicographic order.

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NB: This can also be seen as a lexicographic order.

Proposition

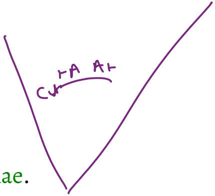
$\prec$ is a well-order on the class of multisets of natural numbers.

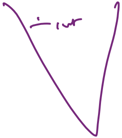
Putting it all together

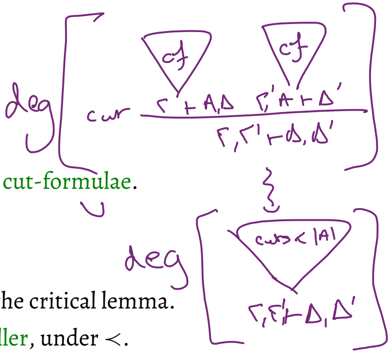
rank
grade

Definition (Degrees)

The **degree** of a proof is the multiset of sizes of its cut-formulae.







Definition (Degrees)

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Proof sketch of cut-elimination.

By induction on the degree of a proof, under $\prec$:

- Identify a topmost cut in a proof, and apply the critical lemma.
- The degree of the resulting proof will be smaller, under $\prec$.
- Thus we may conclude by the inductive hypothesis. □

$\Rightarrow$ reach a NF

$\Rightarrow$ reach a cut-free proof.

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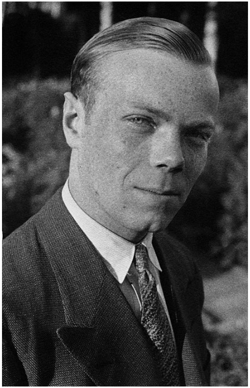
Exercise: fill in all the details for yourself!

Outline

Herbrand's theorem



Theorem (Herbrand '30)

If $\models \exists x.A(x)$, *with* A *quantifier-free, there is a finite set* $\{t_i\}_{i=1}^n$ *s.t.* $\models A(t_1) \vee \cdots \vee A(t_n)$.

$\vdash \exists x A(x) \rightsquigarrow \vdash_{cf} \exists x A(x)$

Proof idea of Herbrand's Theorem

The basic idea is that the terms in the Herbrand disjunction are just those from which the $\exists$ originates.

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Formally, we can prove it by induction on the structure of a cut-free proof.
The interesting cases are:

$$\exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, \exists x.A, \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

The basic idea is that the terms in the Herbrand disjunction are just those from which the $\exists$ originates.

Formally, we can prove it by induction on the structure of a cut-free proof. The interesting cases are:

$$\exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, \exists x.A, \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

Exercise: Try to check how each of the other cases works for yourself!

Digression: recovering richer expressivity

Do we still have cut-free completeness for the richer language with $\vee$ and $\wedge$?

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Definition

We say that a rule is **cut-free-invertible** if, whenever its conclusion is cut-free provable each of its premisses are too.

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Proposition

All of our propositional rules are cut-free-invertible.

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Do we still have cut-free completeness for the richer language with $\vee$ and $\wedge$? It turns out we can recover it directly from the current system!

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We say that a rule is **cut-free-invertible** if, whenever its conclusion is cut-free provable each of its premisses are too.

Proposition

All of our propositional rules are cut-free-invertible.

Proof idea.

By induction on the structure of the proof. Essentially we 'replace' each formula with its 'inverses'. $\square$

Rules for $\vee$, $\wedge$ and an encoding in $\{\to, \bot\}$

Rules for $\vee$, $\wedge$ and an encoding in $\{\rightarrow, \perp\}$

Recall that we can encode $\vee$ and $\wedge$ in $\{\rightarrow, \perp\}$ by adequacy:

$$\begin{array}{ccc} A \vee B & \equiv & \neg A \rightarrow B \\ A \wedge B & \equiv & \neg(A \rightarrow \neg B) \end{array}$$

Rules for $\vee$, $\wedge$ and an encoding in $\{\to, \bot\}$

Recall that we can encode $\vee$ and $\wedge$ in $\{\to, \bot\}$ by adequacy:

$$\begin{array}{ccc} A \vee B & \equiv & \neg A \to B \\ A \wedge B & \equiv & \neg(A \to \neg B) \end{array}$$

We could also have native rules for $\vee$ and $\wedge$:

$$\wedge\text{-}l\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\text{-}r\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee\text{-}r\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

Recall that we can encode $\vee$ and $\wedge$ in $\{\to, \bot\}$ by adequacy:

$$
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A \vee B & \equiv & \neg A \to B \\
A \wedge B & \equiv & \neg(A \to \neg B)
\end{array}
$$

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\qquad
\wedge\text{-}r \, \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}
$$

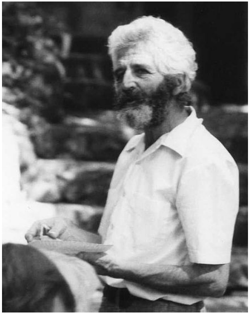
$$
\vee\text{-}r \, \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}
\qquad
\vee\text{-}l \, \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}
$$

Proposition

The sequent system with $\vee$ and $\wedge$ native is also cut-free complete.

Proof idea.

Use cut-free-invertibility of the $\to$ and $\bot$ rules to simulate the rules above. $\square$



Theorem (Craig Interpolation)

If $\models A \to B$*, there is some* I *in the common language of* A *and* B *s.t.* $\models A \to I$ *and* $\models I \to B$.

Proof of Craig Interpolation

By the previous arguments, we will work in the calculus including only the connectives $\neg$, $\vee$, $\wedge$ and all formulae in negation normal form, i.e. with negation only on atoms.

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Proof idea of Craig Interpolation.

By structural induction on a cut-free proof. For instance, if the proof ends with,

$$\wedge\text{-}r\ \dfrac{\begin{array}{c}\pi_0\\ \Gamma \vdash \Delta, A\end{array} \qquad \begin{array}{c}\pi_1\\ \Gamma \vdash \Delta, B\end{array}}{\Gamma \vdash \Delta, A \wedge B}$$

and I_0, I_1 are interpolants obtained from π_0, π_1 resp., then we may define the new interpolant as $I_0 \wedge I_1$. $\square$

Outline

$$\dfrac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ c\text{-}r \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ w\text{-}r$$

LJ: one formula on RHS

$$t \rightsquigarrow_{\beta}^{*} t_1 \cdots\rightsquigarrow^{*} t'$$

$$t \rightsquigarrow_{\beta}^{+} t_0 \cdots\rightsquigarrow^{*} t'$$

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