Introduction to Proof Theory

Lecture 4 - *Hauptsatz*: the cut-elimination theorem

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These slides are available at `http://www.anupamdas.com/mgs21`.

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Recap: the system LK

INITIAL AND STRUCTURAL RULES

$$\text{id}\ \frac{}{A \vdash A} \qquad w\text{-}l\ \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad w\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad c\text{-}l\ \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} \qquad c\text{-}r\ \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}$$

PROPOSITIONAL RULES:

$$\bot\text{-}l\ \frac{}{\bot \vdash} \qquad \bot\text{-}r\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \bot} \qquad \to\text{-}l\ \frac{\Gamma \vdash \Delta, A \quad B, \Gamma \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \qquad \to\text{-}r\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}$$

QUANTIFIER RULES:

$$\forall\text{-}l\ \frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x.A \vdash \Delta} \qquad \forall\text{-}r\ \frac{\Gamma \vdash \Delta, A[y/x]}{\Gamma \vdash \Delta, \forall x.A} \qquad \exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad \exists\text{-}l\ \frac{\Gamma, A[y/x] \vdash \Delta}{\Gamma, \exists x.A \vdash \Delta}$$

where $y \notin \mathsf{FV}(\Gamma, \Delta, A)$.

CUT RULE:

$$cut\ \frac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Outline

The subject matter of *structural proof theory*

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Not only does it yield our much sought after analytic system, it is one of the most powerful tools in all of logic

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"*cut-elimination is the most beautiful result with the most horrible proof*"

This is one of the hardest termination arguments in graduate mathematics and computer science!

The *Hauptsatz*

Theorem (*Hauptsatz*, Gentzen '34)

Every theorem of LK *has a proof that does not use the cut rule.*

Corollary (Subformula property)

Every theorem of LK *has a proof that contains only subformulae of it, up to substitution of variables for terms.*

How cut-elimination works

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Basic idea, in a nutshell

“push” the cuts ***upwards*** *in a proof, until they disappear completely!*

How cut-elimination works

Basic idea, in a nutshell

*"push" the cuts **upwards** in a proof, until they disappear completely!*

I will outline the argument, but understanding comes through practice! I recommend standard references for detailed proofs:

- [Buss, 1998]. *Handbook of Proof Theory*.
- [Troelstra and Schwichtenberg, 1996]. *Basic Proof Theory*.
- [Negri and von Plato, 2001]. *Structural Proof Theory*.
- [Szabo, 1972]. *The Collected Papers of Gerhard Gentzen*.
- Or do it *yourself*!

Outline

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Key propositional case

Key propositional case

$$
\text{cut}\;\dfrac{
\to\text{-}r\;\dfrac{\overset{\triangledown_1}{\Gamma, A \vdash \Delta, B}}{\Gamma \vdash \Delta, A \to B}
\qquad
\to\text{-}l\;\dfrac{\overset{\triangledown_2}{\Gamma' \vdash \Delta', A} \quad \overset{\triangledown_3}{\Gamma', B \vdash \Delta'}}{\Gamma', A \to B \vdash \Delta'}
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

Key propositional case

$$
\dfrac{
\rightarrow\text{-}r\ \dfrac{\overset{\overset{1}{\triangledown}}{\Gamma, A \vdash \Delta, B}}{\Gamma \vdash \Delta, A \rightarrow B}
\qquad
\rightarrow\text{-}l\ \dfrac{\overset{\overset{2}{\triangledown}}{\Gamma' \vdash \Delta', A} \quad \overset{\overset{3}{\triangledown}}{\Gamma', B \vdash \Delta'}}{\Gamma', A \rightarrow B \vdash \Delta'}
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathit{cut}
$$

$$
\rightsquigarrow \quad
c\ \dfrac{
\mathit{cut}\ \dfrac{
\mathit{cut}\ \dfrac{\overset{\overset{2}{\triangledown}}{\Gamma' \vdash \Delta', A} \quad \overset{\overset{1}{\triangledown}}{\Gamma, A \vdash \Delta, B}}{\Gamma, \Gamma' \vdash \Delta, \Delta', B}
\qquad
\overset{\overset{3}{\triangledown}}{\Gamma', B \vdash \Delta'}
}{\Gamma, \Gamma', \Gamma' \vdash \Delta, \Delta', \Delta'}
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

Key quantifier case

Key quantifier case

$$\dfrac{\exists\text{-}r\;\dfrac{\begin{matrix}\vdots\;1\\ \Gamma \vdash \Delta, A(t)\end{matrix}}{\Gamma \vdash \Delta, \exists x.A(x)} \qquad \exists\text{-}l\;\dfrac{\begin{matrix}\vdots\;2(y)\\ \Gamma', A(x) \vdash \Delta'\end{matrix}}{\Gamma', \exists x.A(x) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;cut$$

Key quantifier case

$$\dfrac{\dfrac{\dfrac{\overset{1}{\nabla}}{\Gamma \vdash \Delta, A(t)}}{\Gamma \vdash \Delta, \exists x.A(x)}\ \exists\text{-}r \qquad \dfrac{\dfrac{\overset{2(y)}{\nabla}}{\Gamma', A(x) \vdash \Delta'}}{\Gamma', \exists x.A(x) \vdash \Delta'}\ \exists\text{-}l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{1}{\nabla}}{\Gamma \vdash \Delta, A(t)} \qquad \dfrac{\overset{2(t)}{\nabla}}{\Gamma', A(t) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut$$

Key quantifier case

$$\dfrac{\exists\text{-}r\ \dfrac{\overset{\overset{1}{\triangledown}}{\Gamma \vdash \Delta, A(t)}}{\Gamma \vdash \Delta, \exists x.A(x)} \qquad \exists\text{-}l\ \dfrac{\overset{\overset{2(y)}{\triangledown}}{\Gamma', A(x) \vdash \Delta'}}{\Gamma', \exists x.A(x) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathit{cut} \quad \rightsquigarrow \quad \dfrac{\overset{\overset{1}{\triangledown}}{\Gamma \vdash \Delta, A(t)} \qquad \overset{\overset{2(t)}{\triangledown}}{\Gamma', A(t) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \mathit{cut}$$

- Implicitly assume free variable conditions are respected.
- Write $2(t)$ for the derivation arising from $2(y)$ by replacing every free occurrence of y by t.

Key quantifier case

$$
\dfrac{
\dfrac{\overset{1}{\Gamma \vdash \Delta, A(t)}}{\Gamma \vdash \Delta, \exists x.A(x)}\,\exists\text{-}r
\qquad
\dfrac{\overset{2(y)}{\Gamma', A(x) \vdash \Delta'}}{\Gamma', \exists x.A(x) \vdash \Delta'}\,\exists\text{-}l
}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathit{cut}
\quad \rightsquigarrow \quad
\dfrac{\overset{1}{\Gamma \vdash \Delta, A(t)} \qquad \overset{2(t)}{\Gamma', A(t) \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathit{cut}
$$

- Implicitly assume free variable conditions are respected.
- Write $2(t)$ for the derivation arising from $2(y)$ by replacing every free occurrence of y by t.

NB: The key $\forall$ case is dual.

(Example of) commutative cases

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$$
{\scriptstyle cut}\;\dfrac{\begin{matrix}\overset{1}{\nabla}\\ \Gamma \vdash \Delta, A\end{matrix} \qquad {\scriptstyle \to\text{-}r}\;\dfrac{\begin{matrix}\overset{2}{\nabla}\\ \Gamma', A, B \vdash \Delta', C\end{matrix}}{\Gamma', A \vdash \Delta', B \to C}}{\Gamma, \Gamma' \vdash \Delta, \Delta', B \to C}
\quad \rightsquigarrow \quad
{\scriptstyle \to\text{-}r}\;\dfrac{{\scriptstyle cut}\;\dfrac{\begin{matrix}\overset{1}{\nabla}\\ \Gamma \vdash \Delta, A\end{matrix} \quad \begin{matrix}\overset{2}{\nabla}\\ \Gamma', A, B \vdash \Delta', C\end{matrix}}{\Gamma, \Gamma', B \vdash \Delta, \Delta', C}}{\Gamma, \Gamma' \vdash \Delta, \Delta', B \to C}
$$

$$
{\scriptstyle cut}\;\dfrac{\begin{matrix}\overset{1}{\nabla}\\ \Gamma \vdash \Delta, A\end{matrix} \qquad {\scriptstyle \to\text{-}l}\;\dfrac{\begin{matrix}\overset{2}{\nabla}\\ \Gamma', A \vdash \Delta', B\end{matrix} \quad \begin{matrix}\overset{3}{\nabla}\\ \Gamma', A, C \vdash \Delta'\end{matrix}}{\Gamma', B \to C, A \vdash \Delta'}}{\Gamma, \Gamma', B \to C \vdash \Delta, \Delta'}
$$

$$
\rightsquigarrow \quad
{\scriptstyle \to\text{-}l}\;\dfrac{{\scriptstyle cut}\;\dfrac{\begin{matrix}\overset{1}{\nabla}\\ \Gamma \vdash \Delta, A\end{matrix} \quad \begin{matrix}\overset{2}{\nabla}\\ \Gamma', A \vdash \Delta', B\end{matrix}}{\Gamma, \Gamma' \vdash \Delta, \Delta', B} \qquad {\scriptstyle cut}\;\dfrac{\begin{matrix}\overset{1}{\nabla}\\ \Gamma \vdash \Delta, A\end{matrix} \quad \begin{matrix}\overset{3}{\nabla}\\ \Gamma', A, C \vdash \Delta'\end{matrix}}{\Gamma, \Gamma', C \vdash \Delta, \Delta'}}{\Gamma, \Gamma', B \to C \vdash \Delta, \Delta'}
$$

Outline

Weakening: enter non-confluence

$$\dfrac{\dfrac{\overset{1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A}\,{\scriptstyle w\text{-}r} \qquad \overset{2}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,{\scriptstyle cut} \quad \rightsquigarrow \quad \dfrac{\overset{1}{\Gamma \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,{\scriptstyle w}$$

$$\dfrac{\overset{1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{2}{\Gamma' \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\,{\scriptstyle w\text{-}l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,{\scriptstyle cut} \quad \rightsquigarrow \quad \dfrac{\overset{2}{\Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,{\scriptstyle w}$$

NB: cut-reduction is not confluent!

$$
w\,\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
\quad \leftsquigarrow \quad
\mathit{cut}\,\dfrac{w\text{-}r\,\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A} \qquad w\text{-}l\,\dfrac{\overset{\nabla_2}{\Gamma' \vdash \Delta'}}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
\quad \rightsquigarrow \quad
w\,\dfrac{\overset{\nabla_2}{\Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}
$$

Weakening: enter non-confluence II

NB: cut-reduction is not confluent!

$$
\dfrac{\dfrac{\nabla_1}{\Gamma \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w
\quad \leftsquigarrow \quad
\dfrac{\dfrac{\dfrac{\nabla_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A}\,\textit{w-r} \qquad \dfrac{\dfrac{\nabla_2}{\Gamma' \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\,\textit{w-l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\textit{cut}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{\nabla_2}{\Gamma' \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w
$$

This is known as **Lafont's counterexample**.

Contraction: enter non-termination

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One particularly nasty issue is:

$$
\dfrac{\overset{\overset{1}{\nabla}}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\overset{2}{\nabla}}{\Gamma', A, A \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\,c\text{-}l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,cut
\quad \rightsquigarrow \quad
\begin{array}{c}
\dfrac{\overset{\overset{1}{\nabla}}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\overset{1}{\nabla}}{\Gamma \vdash \Delta, A} \quad \overset{\overset{2}{\nabla}}{\Gamma', A, A \vdash \Delta'}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}\,cut}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}\,cut \\
\overline{\overline{\Gamma, \Gamma' \vdash \Delta, \Delta'}}\,c
\end{array}
$$

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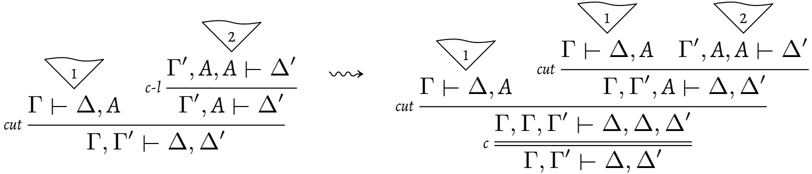

Here the entire subproof 1 is duplicated!

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$$\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\nabla_2}{\Gamma', A, A \vdash \Delta'}}{\Gamma', A \vdash \Delta'}\;c\text{-}l}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;cut \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\nabla_1}{\Gamma \vdash \Delta, A} \quad \overset{\nabla_2}{\Gamma', A, A \vdash \Delta'}}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}\;cut}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}\;cut}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;c$$

Here the entire subproof 1 is duplicated!

NB: the reduction for *c-r* is dual.

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Even worse, just blindly applying permutation rules will not work:

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$$\dfrac{\dfrac{\overset{1}{\nabla} \atop \Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\;\textit{c-r} \qquad \dfrac{\overset{2}{\nabla} \atop \Gamma', A, A \vdash \Delta'}{\Gamma', A \vdash \Delta'}\;\textit{c-l}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\;\textit{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\overset{1}{\nabla} \atop \Gamma \vdash \Delta, A, A \qquad \overset{2}{\nabla} \atop \Gamma', A, A \vdash \Delta'}{\Gamma, \Gamma', A \vdash \Delta, \Delta', A}\;\textit{cut}}{\vdots}$$

The sequent in red is trivial.

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$$
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\quad \rightsquigarrow \quad
\dfrac{\textit{cut}\;\dfrac{\overset{1}{\nabla}\atop{\Gamma \vdash \Delta, A, A} \qquad \overset{2}{\nabla}\atop{\Gamma', A, A \vdash \Delta'}}{\color{red}{\Gamma, \Gamma', A \vdash \Delta, \Delta', A}}}{\vdots}
$$

The sequent in red is trivial.

NB: cut-reduction is not strongly normalising.

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Outline

Lemma

For any proof

$$\textit{cut}\ \frac{\overset{\pi}{\Gamma \vdash \Delta, A} \qquad \overset{\pi'}{\Gamma', A \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \tag{1}$$

where the subproofs π and π' have only cuts on formulas smaller than A, there is a proof of $\Gamma, \Gamma' \vdash \Delta, \Delta'$ with only cuts on formulas smaller than A.

The critical lemma via left-then-right strategy

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Proof idea.

- First push the cut up π' maximally (by induction on $|\pi'|$).
- For any topmost resulting cut, push up the LHS maximally (by induction on $|\mathrm{LHS}|$) and apply key reduction to leave a smaller cut.
- Repeat the process for remaining cuts. □

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- For any topmost resulting cut, push up the LHS maximally (by induction on |LHS|) and apply key reduction to leave a smaller cut.
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More details: ...will be given in the exercise session.

Definition (Multiset measure)

For two multisets M, N of natural numbers, we say $M \prec N$ if I may transform N into M by the continually applying the following operation:

delete a number $n \in N$ and insert any number of numbers $< n$

An induction measure

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Example

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NB: This can also be seen as a lexicographic order.

Proposition

$\prec$ *is a well-order on the class of multisets of natural numbers.*

Putting it all together

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Proof sketch of cut-elimination.

By induction on the degree of a proof, under $\prec$:

- Identify a topmost cut in a proof, and apply the critical lemma.
- The degree of the resulting proof will be smaller, under $\prec$.
- Thus we may conclude by the inductive hypothesis. □

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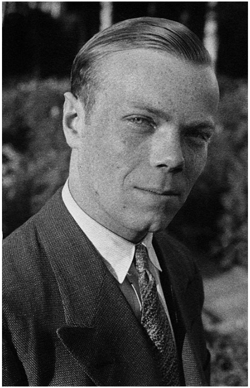
Exercise: fill in all the details for yourself!

Outline



Theorem (Herbrand '30)

If $\models \exists x.A(x)$, *with* A *quantifier-free, there is a finite set* $\{t_i\}_{i=1}^{n}$ *s.t.* $\models A(t_1) \vee \cdots \vee A(t_n)$.

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The basic idea is that the terms in the Herbrand disjunction are just those from which the ∃ originates.

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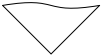
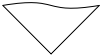


$$\exists\text{-}r\ \frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A}$$



$$c\text{-}r\ \frac{\Gamma \vdash \Delta, \exists x.A, \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

Proof idea of Herbrand's Theorem

The basic idea is that the terms in the Herbrand disjunction are just those from which the ∃ originates.

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The interesting cases are:

$$\exists\text{-}r\,\frac{\Gamma \vdash \Delta, A[t/x]}{\Gamma \vdash \Delta, \exists x.A} \qquad c\text{-}r\,\frac{\Gamma \vdash \Delta, \exists x.A, \exists x.A}{\Gamma \vdash \Delta, \exists x.A}$$

Exercise: Try to check how each of the other cases works for yourself!

Digression: recovering richer expressivity

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We say that a rule is **cut-free-invertible** if, whenever its conclusion is cut-free provable each of its premisses are too.

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All of our propositional rules are cut-free-invertible.

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We say that a rule is **cut-free-invertible** if, whenever its conclusion is cut-free provable each of its premisses are too.

Proposition

All of our propositional rules are cut-free-invertible.

Proof idea.

By induction on the structure of the proof. Essentially we 'replace' each formula with its 'inverses'. □

Rules for $\vee$, $\wedge$ and an encoding in $\{\rightarrow, \perp\}$

Rules for $\vee$, $\wedge$ and an encoding in $\{\rightarrow, \bot\}$

Recall that we can encode $\vee$ and $\wedge$ in $\{\rightarrow, \bot\}$ by adequacy:

$$
\begin{aligned}
A \vee B &\quad\equiv\quad \neg A \rightarrow B \\
A \wedge B &\quad\equiv\quad \neg(A \rightarrow \neg B)
\end{aligned}
$$

Rules for $\vee$, $\wedge$ and an encoding in $\{\rightarrow, \bot\}$

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$$\begin{array}{ccc} A \vee B & \equiv & \neg A \rightarrow B \\ A \wedge B & \equiv & \neg(A \rightarrow \neg B) \end{array}$$

We could also have native rules for $\vee$ and $\wedge$:

$$\wedge\text{-}l\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\text{-}r\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

$$\vee\text{-}r\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

Rules for ∨, ∧ and an encoding in $\{\to, \bot\}$

Recall that we can encode ∨ and ∧ in $\{\to, \bot\}$ by adequacy:

$$\begin{array}{lcl} A \vee B & \equiv & \neg A \to B \\ A \wedge B & \equiv & \neg(A \to \neg B) \end{array}$$

We could also have native rules for ∨ and ∧:

$$\wedge\text{-}l\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \wedge\text{-}r\ \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

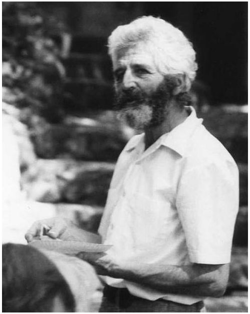
$$\vee\text{-}r\ \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \vee\text{-}l\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}$$

Proposition

The sequent system with ∨ and ∧ native is also cut-free complete.

Proof idea.

Use cut-free-invertibility of the → and ⊥ rules to simulate the rules above. □



Theorem (Craig Interpolation)

If $\models A \to B$*, there is some* I *in the common language of* A *and* B *s.t.* $\models A \to I$ *and* $\models I \to B$.

Proof of Craig Interpolation

By the previous arguments, we will work in the calculus including only the connectives $\neg$, $\vee$, $\wedge$ and all formulae in negation normal form, i.e. with negation only on atoms.

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NB: This form means that formulas do not change sides. Beware, this is not always possible for other logics!

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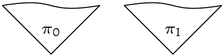
Proof idea of Craig Interpolation.

By structural induction on a cut-free proof. For instance, if the proof ends with,



$$\wedge\text{-}r\ \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$$

and I_0, I_1 are interpolants obtained from π_0, π_1 resp., then we may define the new interpolant as $I_0 \wedge I_1$. □

Outline

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