Gentzen proved Hauptsatz for classical and intuitionistic predicate logic.

- Analyticity
- Decidability of propositional fragments
  ↳ non-provability methods
- Witness Extraction (Herbrand)
- Separability algorithms (Craig interpolation)

Metamathematical motivations

Gödel: $PA \nvdash Con_{PA}$

Gentzen '38: $Con_{PA} \iff$ well-foundedness of $\varepsilon_0 = \omega^{\omega^{\omega^{\cdots}}}$.
(using cut-elimination)

$$\dfrac{\boxed{\dfrac{\Gamma \vdash A(0) \quad \vdash A(1) \quad \vdash A(2) \quad \cdots}{\Gamma \vdash \forall x A}} \qquad \dfrac{\Gamma, A(u) \vdash \Delta}{\Gamma, \forall x A \vdash \Delta}}{\Gamma \vdash \Delta}$$

$$\rightsquigarrow \quad \dfrac{\Gamma \vdash A(a) \qquad \Gamma, A(u) \vdash \Delta}{\Gamma \vdash \Delta}$$

Corollary of Hauptsatz: LK is consistent.

Proof: otherwise there would be a proof of $\vdash$
$\implies \exists$ cut-free proof of $\vdash$
But no cut-free rule concludes with $\vdash$. ✗

Method extended to $\boxed{\Pi^1_1\text{-CA} \rightsquigarrow ID_{<\omega}}$
$\rightsquigarrow$ PA2 (impredicative) (Takeuti's conjecture)
(Tait, Prawitz, Takahashi '6x)

Scalability

- Substructural logics. Relevant or Linear logics.
- Non-commutative logics. Lambek calculus, algebraic logics.
- Modal logics. K, S4, and some constructive variants.
- (second-order logic)

Modern proof theory:
- has more structure (hyper, labelled, nested) sequents, cyclic proofs
- is more compositional (deep inference, categorical logic, proof nets)
- is more symmetric (display calculus, deep inference)

Computational content of proofs

Witnessing: From a proof of $\forall x \exists y A(x,y)$ extract a program f s.t. $\models \forall x A(x, f(x))$

Claim: $\exists a, b \notin \mathbb{Q}.\ a^b \in \mathbb{Q}$

Proof: $\sqrt{2} \notin \mathbb{Q}$ so consider $\sqrt{2}^{\sqrt{2}}$.
Case 1. $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$. ✓
Case 2. $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$. $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$ ✓

$\{ (\sqrt{2}, \sqrt{2}), \boxed{(\sqrt{2}^{\sqrt{2}}, \sqrt{2})} \}$

↳ cannot always extract computational content from classical proofs.

Observation: Suppose $\boxed{\mathbb{N} \models \forall x \exists y \underbrace{A(x,y)}_{\text{recursive}}}$.
Then there is a program f s.t. $\models \forall x A(x, f(x))$

Kreisel: "what more can we know from a proof of a theorem other than the fact that it is true"

- $PA \vdash \forall x \exists y A(x,y) \implies$ there is a term t of system T s.t. $T \vdash \forall x A(x, t(x))$
- $\underbrace{I\Sigma_1}_{\text{induction on semi-recursive predicates}} \vdash \forall x \exists y A(x,y) \implies \ldots t$ primitive recursive $\ldots$ PRA $\vdash \ldots$

Curry-Howard(-Lambek) correspondence

Gentzen's Motivation

$$\begin{tikzcd} ND \arrow[r, "easy"] \arrow[d, dotted, "normalisation"'] & LK \arrow[d, "cut\text{-}elim."] \\ \text{normal } ND & \text{cut-free } LK \arrow[l, "easy"] \end{tikzcd}$$

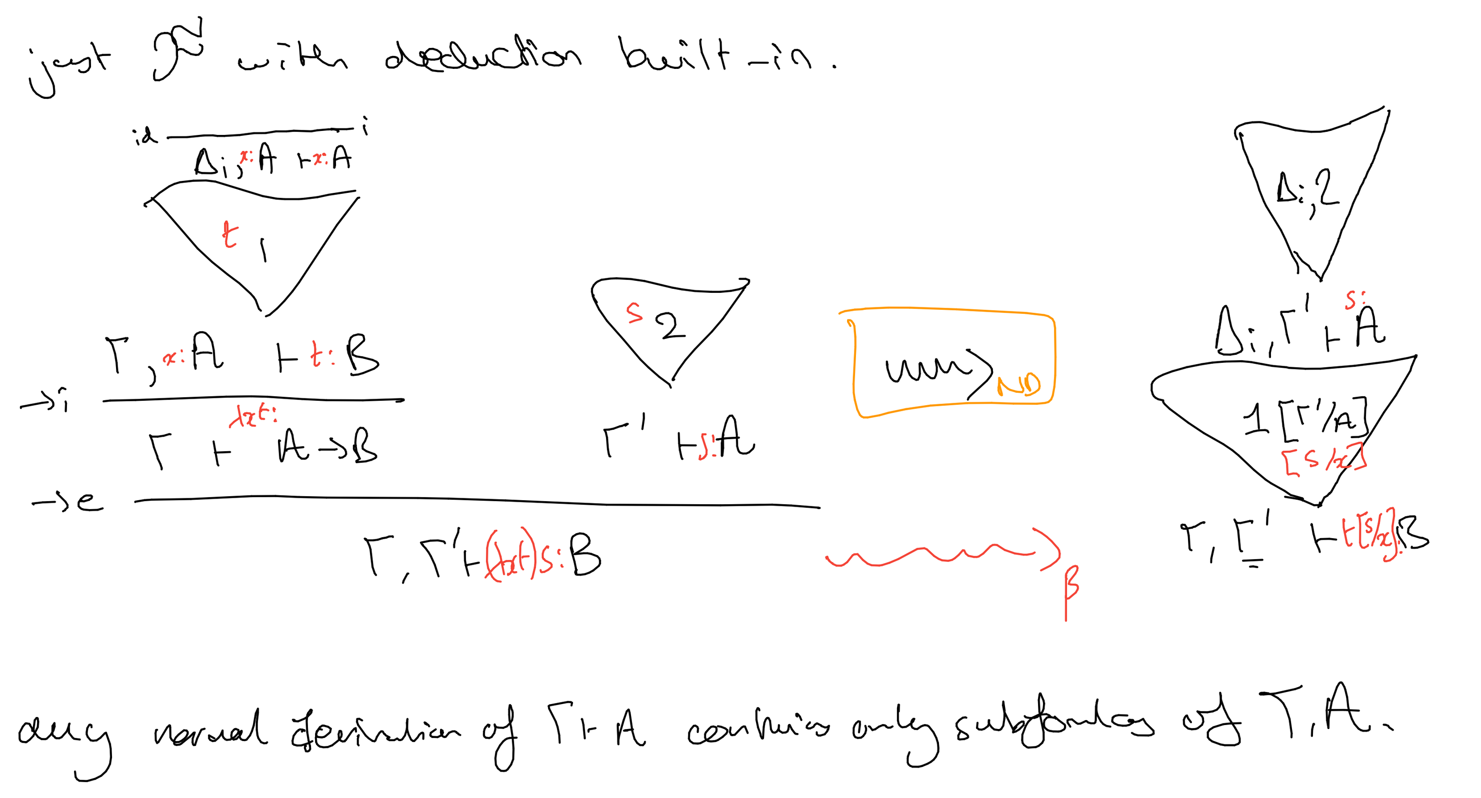
$$\text{cut}\ \dfrac{\overline{A \vdash A} \qquad \overline{A \vdash A}}{A \vdash A} \quad \rightsquigarrow \quad \overline{A \vdash A}$$

Gentzen's Natural Deduction ($ND_{\to}$)

HF — Deduction Theorem

$$\text{id}\ \dfrac{}{\Gamma, x{:}A \vdash x{:}A} \qquad \to\text{-e}\ \dfrac{\Gamma \vdash t{:}A \to B \quad \Gamma' \vdash s{:}A}{\Gamma, \Gamma' \vdash ts{:}B} \qquad \to\text{-i}\ \dfrac{\Gamma, x{:}A \vdash t{:}B}{\Gamma \vdash \lambda x t{:}A \to B}$$

ND is just HF with deduction built-in.



$\implies$ any normal derivation of $\Gamma \vdash A$ contains only subformulas of Γ, A.

The Correspondence

$ND_{\to}$	$\lambda_{\to}$
Formulas	Types
introduction / elimination	constructors / destructors
proofs	terms
normalisation	execution

$\implies$ $\rightsquigarrow_{ND}$ is confluent and strongly normalising