Proof theory of arithmetic

Lecture 1 – The rise of proof theory

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These slides are available at `http://www.anupamdas.com/esslli23`.
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Outline

Wherefore proof theory?

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Formally, a **proof system** defines what a proof may be. This allows not only the study of what is provable, but also of proofs themselves.

But *why* do we study proofs?

- *Foundational results*, which tell us something fundamental about reasoning itself. These lead to...
- *Applications*, which use insights and techniques from proof theory to accomplish something concrete in another discipline altogether.

Foundational results from the early days

Gödel's incompleteness theorem (1931), informally

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Gentzen's *Hauptsatz* (1934), informally

Every purely logical theorem can be proved analytically, i.e. without making 'guesses'.

The translation of proofs to analytic form, cut-elimination, has become one of the most powerful tools in all of logic.

Proof theory and computation

Proof theory plays an essential role in understanding computation:

- Extracting *computational content* via normal forms. (cut-elimination, normalisation,…)
- …and by proof interpretations. (realisability, Dialectica, witnessing,…)
- Modelling non-deterministic complexity (proof complexity, proof search,…)
- Proof normalisation as computation. (Curry-Howard-Lambek,…)
- Proof search as computation. (focussing, logic programming,…)

References for this course

There are a number of excellent textbooks, including but not limited to:

1. [Takeuti, 1975] *Proof Theory*.
2. [Pohlers, 1989] *Proof Theory: an Introduction*.
3. [Buss, 1998] *Handbook of Proof Theory*.
4. [Troelstra and Schwichtenberg, 1996] *Basic Proof Theory*.
5. [Arai, 2020] *Ordinal Analysis with an Introduction to Proof Theory*.

For excellent high-level references, see also:

6. *Stanford Encyclopedia of Philosophy*. `http://plato.stanford.edu/`

NB: While 1 and 2 are a little dated, they remain seminal introductions to the field.

Sprechen Sie deutsch?

For this lecture, I recommend taking a look at original texts:

- [Frege, 1879] *Begriffsschrift*. (The first serious attempt to formalise mathematics in logic, unfortunately erroneously)
- [Whitehead and Russell, 1927] *Principia Mathematica*. (The second serious attempt, notably error-free)
- [Gödel, 1931] *Über Formal Unentscheidbare Sätze der Principia Mathematica Und Verwandter Systeme I*. (The incompleteness theorems)
- [Szabo, 1972] *The collected papers of Gerhard Gentzen*. English translations. (The foundations of modern proof theory)

Outline

"Concrete" mathematics

Up to the 19^{th} century, mathematics was primarily concerned with concrete objects which could be explicitly constructed. E.g.:

- Every number is either even or odd.
- There are infinitely many prime numbers.
- Every number can be written as the sum of four squares.
- Every continuous function can be integrated.
- Every non-constant polynomial over over the complex numbers has a root.

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Proofs of these results yield concrete algorithms. E.g.:

- We can decide whether a number is even or odd.
- We can find the next prime number.
- For any number we can find four squares which sum to that number.
- We can compute the integral of f up to *any desired accuracy*.
- We can compute roots of polynomials up to *any desired accuracy*.

“Non-constructive” mathematics

With the advent of modern mathematics in the 19^{th} century, mathematicians started to reason about non-constructible objects.

- For every $f : \mathbb{N} \to \mathbb{N}$ there exists an n such that $f(n) \leqslant f(m)$ for all m.
- Every set of real numbers has a least upper bound.
- Every monotone, bounded sequence converges to a limit.
- Every ring has a maximal ideal.
- Every vector space has a basis.

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Question

*Do these ideal objects really '*exist*'?*

Example: irrational powers

Proposition

There are irrational numbers a, b such that a^b is rational.

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Proof.

We know that $\sqrt{2}$ is irrational. What about $\sqrt{2}^{\sqrt{2}}$? We have two cases:

- If $\sqrt{2}^{\sqrt{2}}$ is rational, then set $a = b = \sqrt{2}$.
- Otherwise $\sqrt{2}^{\sqrt{2}}$ is irrational. Set $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. We have,

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2$$

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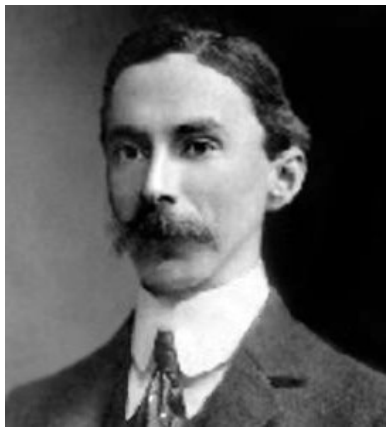
Question: Is $\sqrt{2}^{\sqrt{2}}$ rational or irrational?!

Down the rabbit hole...

Even worse, this style of mathematical reasoning can lead us down fallacious paths.



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- If $R \in R$ then by definition we must have that $R \notin R$;
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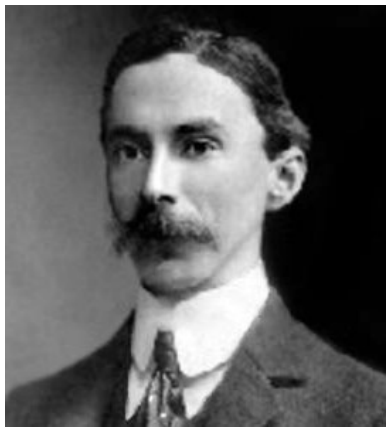

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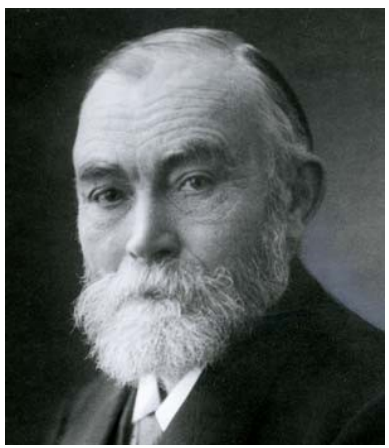
Conclusion: Our naive formulation of mathematics, in particular set theory, is *inconsistent*: it cannot be trusted.

This one simple paradox destroyed a life's work:

Corollary

G. Frege's foundations of arithmetic are inconsistent.



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The tragedy of Frege's foundations led to the so-called **foundational crisis**:

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- What *is* a mathematical proof?
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A HISTORICAL NOTE: Set-theoretic paradoxes were arguably known already by the 1880s, e.g. the *Burali-Forti* paradox. However, the sheer simplicity of Russell's paradox shook the mathematical world, questioning the most basic principles of mathematical reasoning.

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We follow the formalist approach, but both have had deep impact on foundations.

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Hilbert's Program, 1921

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Aside: But this statement is circular! To show that P is consistent we need to work in some other system, which in turn needs to be shown to be consistent etc. Hilbert was well aware of this, so he asked for the following further refinement:

(continued)

3. **Finitary consistency:** the fact that P is consistent is demonstrable using only simple **finitary** methods, whose validity *cannot be questioned*.

Unwinding Hilbert's formalism

The notion of '*finitary*' in Hilbert's program is crucial (and controversial). It distinguishes object-level systems P from 'finitary' meta-level systems N:

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Example: Hilbert's formalist philosophy identifies:

- Limits of monotone bounded sequences exist;
- There is a proof in P that every monotone bounded sequence has a limit;

for an appropriate system P, in particular satisfying (1), for an appropriate system N.

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ENTER K. GÖDEL:

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References for this part: [Raatikainen, 2018, Smullyan, 1992].

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Thus P *does not prove* every true sentence, yielding first incompleteness.

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Idea of argument: Formalise proof of the first incompleteness within P itself.

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(nec) *If* $\mathsf{P} \vdash \varphi$ *then* $\mathsf{P} \vdash \Box\varphi$.

(k) $\mathsf{P} \vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

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Theorem (Löb, 1955)

If $\mathsf{P} \vdash \Box\varphi \to \varphi$ *then* $\mathsf{P} \vdash \varphi$.

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This has a purely modal proof (using conditions above), after just one instance of the *Diagonal Lemma*, giving ψ such that:

$$\mathsf{P} \vdash \psi \leftrightarrow (\Box\psi \to \varphi) \tag{2}$$

Concluding second incompleteness

From *Löb's theorem* we immediately arrive at *Gödel's second incompleteness*:

Corollary (Gödel's Second Incompleteness Theorem, again)

If P *proves its own consistency, then* P *is, in fact, inconsistent*.

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Proof.

Writing $\mathrm{Con}(\mathsf{P})$ for $\neg\Box\bot$ ("there is no proof of a contradiction"), we have:

$$\begin{array}{lll} \mathsf{P} \vdash \mathrm{Con}(\mathsf{P}) & \Longrightarrow & \mathsf{P} \vdash \Box\bot \to \bot \quad \text{by pure logic} \\ & \Longrightarrow & \mathsf{P} \vdash \bot \quad \text{by Löb's theorem.} \end{array} \quad \square$$

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Hilbert's program, in its strictest sense, is unachievable.

Outline

Exercises

1. Use the *Diagonal Lemma* to derive Tarski's *Undefinability of Truth:* there is no formula $\mathrm{Tr}(x)$ of arithmetic such that $\mathsf{P} \vdash \mathrm{Tr}(\ulcorner\psi\urcorner) \leftrightarrow \psi$, for any sentence ψ.
2. Show that the following three formulations of 'consistency' are equivalent:
   a. P does not prove φ and $\neg\varphi$, for any φ.
   b. P does not prove $\bot$.
   c. There is something that P does not prove.
3. What does it mean to be a fixed point (i.e. the sentence ψ from the *Diagonal Lemma*) of the formula $\mathrm{Even}(x)$, stating "x is even".
4. **(Hard)** Prove *Löb's Theorem* using only the 'modal' HBL conditions from just the one instance of the *Diagonal Lemma* in (2).

   Hint: it is known that *Löb's Theorem* cannot be proved in this way without using the *contraction*: from $\varphi \to \varphi \to \psi$ infer $\varphi \to \psi$.
5. (Think about how to) construct an appropriate injection $\ulcorner\cdot\urcorner$: from logical syntax to $\mathbb{N}$. Reflect on what arithmetic you need to (de)code.

Answers to exercises I

1. Suppose otherwise, and use the Diagonal Lemma to construct a sentence ψ such theat $\mathsf{P} \vdash \psi \leftrightarrow \neg \mathrm{Tr}(\ulcorner \psi \urcorner)$. Now, by plugging the two equivalences together, we get $\mathsf{P} \vdash \mathrm{Tr}(\ulcorner \psi \urcorner) \leftrightarrow \neg \mathrm{Tr}(\ulcorner \psi \urcorner)$, which contradicts consistency of P.
2.
   - (b) $\Longrightarrow$ (a). By contraposition, if P proves both φ and $\neg\varphi$, which is equivalent to $\varphi \to \bot$, then $\mathsf{P} \vdash \bot$ simply by *modus ponens*.
   - (c) $\Longrightarrow$ (b). By contraposition, if $\mathsf{P} \vdash \bot$ then $\mathsf{P} \vdash \varphi$ for any φ by *ex falso quodlibet*.
   - (a) $\Longrightarrow$ (c). By contraposition, and choose any formula φ.
3. ψ is a fixed point of $\mathrm{Even}(x)$ just if:
   - $\mathsf{P} \vdash \psi$ and $\ulcorner \psi \urcorner$ is even; or,
   - $\mathsf{P} \vdash \neg\psi$ and $\ulcorner \psi \urcorner$ is odd.
4. Assume $\mathsf{P} \vdash \Box\varphi \to \varphi$, and by the Diagonal Lemma let ψ be a sentence with $\mathsf{P} \vdash \psi \leftrightarrow (\Box\psi \to \varphi)$. We derive φ in P using only HBL conditions as follows:

$\psi \to (\Box\psi \to \varphi)$	by (2)($\to$)	$\Box\psi \to \Box\varphi$	by pure logic
$\Box(\psi \to (\Box\psi \to \varphi))$	by (nec)	$\Box\psi \to \varphi$	by assumption
$\Box\psi \to \Box(\Box\psi \to \varphi)$	by (k)	ψ	by (2)($\leftarrow$)
$\Box\psi \to (\Box\Box\psi \to \Box\varphi)$	by (k)	$\Box\psi$	by (nec)
$\Box\psi \to (\Box\psi \to \Box\varphi)$	by (4)	φ	by *mp*.

Outline

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The trustworthyness of P is reduced to that of 'simple' computational principles.

Computational interpretations

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The spirit of Hilbert's program has led to deep connections between:

- Induction on complex invariants (quantifiers over $\mathbb{N}$, $\mathbb{R}$,...);
- Recursion up to explicit ordinals. (e.g. Gentzen's cut-elimination);
- Recursion at higher types. (e.g. Gödel *Dialectica*, Kleene/Kreisel *realisability*).

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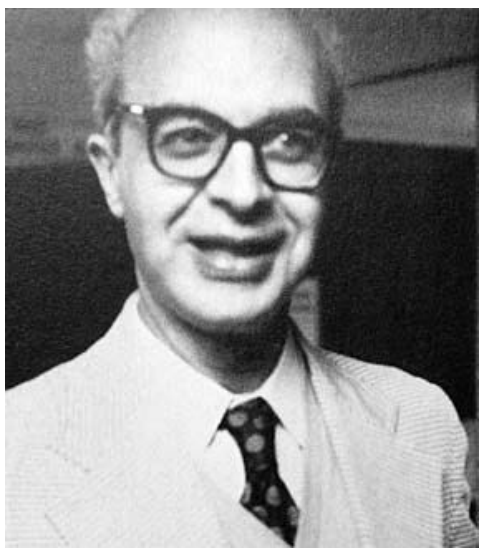
These have lead to one of the most successful applications of logic to mathematics:

Kreisel's unwinding program: **proof mining**



"What more do we know if we have proved a theorem by restricted mean than if we merely know that it is true?"

Outline

Goal of this course

This lecture course will focus on the connections between induction in arithmetic and recursion on higher ordinals, in the spirit of G. Gentzen.

We emphasise mathematical methods, but only sketch formalisations and explicit ordinal calculations.

Structure of this course

Background and motivation

Today: *The rise of proof theory*. Foundational crisis, Hilbert's program, Gödel incompleteness.

In-depth exploration

Tuesday: *Peano Arithmetic*. Syntax of first-order arithmetic, semantics, inductive reasoning, sequent calculus.

Wednesday: *Two consistency proofs*. Soundness (formalised), an infinitary system, cut-elimination, ordinal bounds.

Thursday: *Provably recursive functions*. Ordinal notations, transfinite recursion, Kleene realisability (time permitting).

Overview

Friday: *Further directions*. Fragments of arithmetic, extensions of arithmetic, second-order arithmetic.

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