PROOF THEORY OF ARITHMETIC

# Lecture 3 – A consistency proof


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# Recap: the state of play

We want to reduce the consistency of PA to the consistency of a simple computational theory, in a finitistic manner.

We have formulated PA in sequent style, $\vdash_{\mathsf{PA}}$, and shown their equivalence finitistically, i.e. in PRA.

Almost every rule of $\vdash_{\mathsf{PA}}$ has the **subformula property** (up to substitutions): the only barrier to proving consistency is the cut rule:

$$\mathsf{cut}\ \frac{\vdash_{\mathsf{PA}} \Gamma, \varphi \qquad \vdash_{\mathsf{PA}} \Gamma, \overline{\varphi}}{\vdash_{\mathsf{PA}} \Gamma, \Gamma'}$$

**NB:** this viewpoint arguably hides the real culprit: induction.

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# Outline

# System $\vdash_\omega$ (aka $\mathsf{PA}_\omega$)

We define a judgement $\vdash_\omega$ just like $\vdash_{\mathsf{PA}}$ except:

- Instead of id, $=$, $\neq$, Q rules, we simply admit all closed true (in)equations:

$$= \frac{}{\vdash_\omega n = n} \qquad \neq \frac{}{\vdash_\omega m \neq n}\, m \neq n$$

- Instead of $\forall$, ind rules, we have an $\omega$-branching rule:

$$\omega \frac{\vdash_\omega \Gamma, \varphi(0) \quad \vdash_\omega \Gamma, \varphi(1) \quad \vdash_\omega \Gamma, \varphi(2) \quad \cdots}{\vdash_\omega \Gamma, \forall x \varphi(x)}$$

## Conventions

- $\vdash_\omega$ is a judgement on only closed sequents.
- We *identify* closed terms $t$ with their values $t^{\mathfrak{N}}$ (which are also numerals).

  **NB:** this is subtle, and important for formalisation; we will revisit it later.

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**Definition**

The judgement $\vdash_\omega \Gamma$, for $\Gamma$ closed, is defined by:

$$= \frac{}{\vdash_\omega n = n} \qquad \neq \frac{}{\vdash_\omega m \neq n}\, m \neq n$$

$$\mathsf{w}\ \frac{\vdash_\omega \Gamma}{\vdash_\omega \Gamma, \Gamma'} \qquad \mathsf{cut}\ \frac{\vdash_\omega \Gamma, \varphi \quad \vdash_\omega \Gamma', \overline{\varphi}}{\vdash_\omega \Gamma, \Gamma'}$$

$$\vee\ \frac{\vdash_\omega \Gamma, \varphi_0 \vee \varphi_1, \varphi_i}{\vdash_\omega \Gamma, \varphi_0 \vee \varphi_1}\, i \in \{0, 1\} \qquad \wedge\ \frac{\vdash_\omega \Gamma, \varphi \quad \vdash_\omega \Gamma, \psi}{\vdash_\omega \Gamma, \varphi \wedge \psi}$$

$$\exists\ \frac{\vdash_\omega \Gamma, \exists x \varphi(x), \varphi(n)}{\vdash_\omega \Gamma, \exists x \varphi(x)} \qquad \omega\ \frac{\vdash_\omega \Gamma, \varphi(0) \quad \vdash_\omega \Gamma, \varphi(1) \quad \vdash_\omega \Gamma, \varphi(2) \quad \cdots}{\vdash_\omega \Gamma, \forall x \varphi(x)}$$

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# Example: general identity

Similarly to $\vdash_{\mathsf{PA}}$, we can derive general identity:

**Proposition (General identity)**

$\vdash_\omega \varphi, \overline{\varphi}$

**Proof.**

By induction on the structure of $\varphi$.
The critical difference to the argument for $\vdash_{\mathsf{PA}}$ is for a quantified formula:

$$\omega \dfrac{\exists \dfrac{\overset{IH}{\vdash_\omega \overline{\varphi}(0), \varphi(0)}}{\vdash_\omega \overline{\varphi}(0), \exists x \varphi(x)} \quad \exists \dfrac{\overset{IH}{\vdash_\omega \overline{\varphi}(1), \varphi(1)}}{\vdash_\omega \overline{\varphi}(1), \exists x \varphi(x)} \quad \cdots}{\vdash_\omega \forall x \overline{\varphi}(x), \exists x \varphi(x)}$$

□

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# Example: consistency

Write $\mathrm{Prf}(x, y)$ for a formula stating "$x$ codes a PA-proof of the formula coded by $y$".

It is not hard to see that:

**Proposition ($\vdash_\omega$ *partial* consistency of PA)**

*For each $n \in \mathbb{N}$, we have $\vdash_\omega \neg \mathrm{Prf}(n, \ulcorner \bot \urcorner)$.*

I.e. no *actual* proof concludes $\bot$ (of course, assuming PA is consistent).

However from here we immediately have from the $\omega$ rule:

**Corollary ($\vdash_\omega$ consistency of PA)**

$\vdash_\omega \forall x \neg \mathrm{Prf}(x, \ulcorner \bot \urcorner)$.

**NB:** there was nothing special about PA! We could have started from PA2, ZFC,...

# Beware of god-mode

$\vdash_\omega$ is not effectively presented, and so Gödel incompleteness does not apply.

Indeed we have achieved extreme power:

## Proposition (Completeness)

*If* $\mathfrak{N} \models \varphi$ *then* $\vdash_\omega \varphi$.

**NB:** this argument does not even need cuts!

## Proof idea.

By induction on the structure of $\varphi$. □

Just to be clear, we have a sanity check:

## Proposition (Soundness)

*If* $\vdash_\omega \Gamma$ *then* $\mathfrak{N} \models [\Gamma]$.

## Proof idea.

By induction on $\vdash_\omega$. □

**NB:** Obviously, this slide, and the previous one, work under a strong metatheory.

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# Outline

To organise our overarching induction argument, let us set up some machinery.

### Definition (Degrees)

- The **degree** of a formula $\varphi$, written $\deg(\varphi)$, is its number of logical symbols.
- The degree of a cut step is the degree of its cut formula.
- A $d$-cut is a cut of degree $d$.

Write $\vdash_{\overline{d}}$ for the restriction of $\vdash_{\omega}$ to only $< d$-cuts.

**NB:** Implicit here is that $\deg(\varphi) = \deg(\overline{\varphi})$.

**NB:** $\vdash_{\overline{0}}$ is just cut-free provability. Thus we have (in PRA):

### Proposition (Cut-free consistency)

$\nvdash_{\overline{0}}$

The main result of this section is:

**Theorem**

*If* $\vdash_{\mathsf{PA}} \varphi$, *for* $\varphi$ *closed, then* $\vdash_{d}^{\omega} \varphi$, *for some* $d < \omega$.

More concretely, we shall prove the following Lemma:

**Lemma (Simulation)**

*If* $\vdash_{\mathsf{PA}} \Gamma(\vec{x})$, *with all free variables displayed, then* $\vdash_{d}^{\omega} \Gamma(\vec{n})$ *for all* $\vec{n} \in \mathbb{N}$, *for some* $d < \omega$.

**Proof.**

By induction on $\vdash_{\mathsf{PA}}$. The cases are given over the next few slides.

**NB:** we will typically suppress the substitution of numerals $\vec{n}$ for free variables $\vec{x}$.

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# Simulation of a cut

$$\text{cut}\ \frac{\vdash_{\mathsf{PA}} \Gamma, \varphi \qquad \vdash_{\mathsf{PA}} \Gamma', \overline{\varphi}}{\vdash_{\mathsf{PA}} \Gamma, \Gamma'} \quad \rightsquigarrow \quad \text{cut}\ \frac{\overset{\nabla_{IH}}{\vdash_{\overline{d}} \Gamma, \varphi} \qquad \overset{\nabla_{IH}}{\vdash_{\overline{d}} \Gamma', \overline{\varphi}}}{\vdash_{\overline{e}} \Gamma, \Gamma'}$$

where $\deg(\varphi) = d'$ and $e := \max(d, d')$.

# Simulation of quantifier rules

$$\exists\,\frac{\vdash_{\mathsf{PA}} \Gamma, \exists x \varphi(x), \varphi(t)}{\vdash_{\mathsf{PA}} \Gamma, \exists x \varphi(x)} \quad \rightsquigarrow \quad \exists\,\frac{\overset{IH}{\vdash_{d} \Gamma, \exists x \varphi(x), \varphi(t^{\mathfrak{N}})}}{\vdash_{d} \Gamma, \exists x \varphi(x)}$$

$$\forall\,\frac{\vdash_{\mathsf{PA}} \Gamma, \varphi(y)}{\vdash_{\mathsf{PA}} \Gamma, \forall x\, \varphi(x)} \quad \rightsquigarrow \quad \omega\,\frac{\overset{IH}{\vdash_{d} \Gamma, \varphi(0)} \quad \overset{IH}{\vdash_{d} \Gamma, \varphi(1)} \quad \overset{IH}{\vdash_{d} \Gamma, \varphi(2)} \quad \cdots}{\vdash_{d} \Gamma, \forall x\, \varphi(x)}$$

**NB:** notice here the utility of identifying closed terms with their values. This will facilitate the corresponding 'cut-reduction'.

$$\textsf{ind}\ \frac{\vdash_{\mathsf{PA}} \Gamma, \varphi(0) \qquad \vdash_{\mathsf{PA}} \Gamma, \overline{\varphi}(y), \varphi(\mathsf{s}y)}{\vdash_{\mathsf{PA}} \Gamma, \varphi(t)}$$

Write $d' := \deg(\varphi(x))$, and set $e := \max(d, d')$.

We show that $\vdash_e \Gamma, \varphi(n)$ by (sub)induction on $n$:

- $\vdash_d \Gamma, \varphi(0)$ by *IH*, and so $\vdash_e \Gamma, \varphi(0)$ since $d \leqslant e$.
- For the inductive step we have:

$$\textsf{cut}\ \frac{\overset{ih}{\vdash_e \Gamma, \varphi(n)} \qquad \overset{IH}{\vdash_d \Gamma, \overline{\varphi}(n), \varphi(n+1)}}{\vdash_e \Gamma, \varphi(n+1)}$$

From here we indeed obtain $\vdash_e \Gamma, \varphi(t^{\mathfrak{N}})$, as required.

$$\mathsf{ind}\ \frac{\vdash_{\mathsf{PA}} \Gamma, \varphi(0) \qquad \vdash_{\mathsf{PA}} \Gamma, \overline{\varphi}(y), \varphi(\mathsf{s}y)}{\vdash_{\mathsf{PA}} \Gamma, \varphi(t)}$$

$$\rightsquigarrow \quad \mathsf{cut}\ \dfrac{\mathsf{cut}\ \dfrac{\mathsf{cut}\ \dfrac{\mathsf{cut}\ \dfrac{\mathsf{cut}\ \dfrac{\overset{\mathit{IH}}{\vdash_d \Gamma, \varphi(0)} \qquad \overset{\mathit{IH}}{\vdash_d \Gamma, \overline{\varphi}(0), \varphi(1)}}{\vdash_e \Gamma, \varphi(1)}}{\ddots}}{\vdash_e \Gamma, \varphi(t^{\mathfrak{N}} - 2)} \qquad \overset{\mathit{IH}}{\vdash_d \Gamma, \overline{\varphi}(t^{\mathfrak{N}} - 2), \varphi(t^{\mathfrak{N}} - 1)}}{\vdash_e \Gamma, \varphi(t^{\mathfrak{N}} - 1)} \qquad \overset{\mathit{IH}}{\vdash_d \Gamma, \overline{\varphi}(t^{\mathfrak{N}} - 1), \varphi(t^{\mathfrak{N}})}}{\vdash_e \Gamma, \varphi(t^{\mathfrak{N}})}$$

# Outline

Our main result is:

**Theorem (Cut-Elimination)**
*If* $\vdash_{\overline{d}} \Gamma$ *then we can derive* $\vdash_0 \Gamma$*, i.e. we can derive* $\Gamma$ *in* $\vdash_\omega$ *without using the* cut *rule.*

By the subformula property of the non-cut rules, it is immediate that:

**Corollary (Consistency of $\vdash_{\overline{d}}$)**
$\nvdash_{\overline{d}}$ *, for all* $d < \omega$.

Since we have reduced $\vdash_{\mathsf{PA}}$ to $\bigcup\{\vdash_{\overline{d}}\}_{d<\omega}$, and also $\mathsf{PA} \vdash$ to $\vdash_{\mathsf{PA}}$, we thus have:

**Corollary (Consistency of PA)**
PA *is consistent.*

**NB:** assuming *Cut-Elimination*, the corollaries are obtained finitistically, i.e. in PRA.

First, let us see some cut-reduction cases...

# Cut-reduction cases: commutative

$$\dfrac{\mathsf{r}\,\dfrac{\{\vdash_{\overline{d}} \Gamma_i, \varphi\}_{i<\iota}}{\vdash_{\overline{d}} \Gamma, \varphi} \qquad \vdash_{\overline{d}} \Gamma', \overline{\varphi}}{\vdash_{\overline{d}} \Gamma, \Gamma'}\,\mathsf{cut} \quad \rightsquigarrow \quad \mathsf{r}\,\dfrac{\left\{ \mathsf{cut}\,\dfrac{\vdash_{\overline{d}} \Gamma_i, \varphi \qquad \vdash_{\overline{d}} \Gamma', \overline{\varphi}}{\vdash_{\overline{d}} \Gamma_i, \Gamma'} \right\}_{i<\iota}}{\vdash_{\overline{d}} \Gamma, \Gamma'}$$

for an $\iota$-ary inference step $\mathsf{r}$, with $\iota \in \{0, 1, 2, \omega\}$.

# Cut-reduction case: quantifiers

$$\dfrac{\exists\,\dfrac{\vdash_{\overline{d}} \Gamma', \exists x\varphi(x), \varphi(k)}{\vdash_{\overline{d}} \Gamma', \exists x\varphi(x)} \qquad \omega\,\dfrac{\{\vdash_{\overline{d}} \Gamma, \overline{\varphi}(n)\}_{n<\omega}}{\vdash_{\overline{d}} \Gamma, \forall x\overline{\varphi}(x)}}{\vdash_{\overline{d}} \Gamma, \Gamma'}\ \textsf{cut}$$

$$\rightsquigarrow \quad \dfrac{\textsf{cut}\,\dfrac{\vdash_{\overline{d}} \Gamma', \exists x\varphi(x), \varphi(k) \qquad \omega\,\dfrac{\{\vdash_{\overline{d}} \Gamma, \overline{\varphi}(n)\}_{n<\omega}}{\vdash_{\overline{d}} \Gamma, \forall x\overline{\varphi}(x)}}{\vdash_{\overline{d}} \Gamma, \Gamma', \varphi(k)} \qquad \vdash_{\overline{d}} \Gamma, \overline{\varphi}(k)}{\vdash_{\overline{d}} \Gamma, \Gamma'}\ {<}d\text{-}\textsf{cut}$$

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# Cut-elimination: the big idea, visually

**Lemma ($d$-cut-admissibility)**

$\vdash_{\overline{d}}$ *is closed under* $d$-cut.

*I.e., when* $\deg(\varphi) = d$*, if* $\vdash_{\overline{d}} \Gamma, \varphi$ *and* $\vdash_{\overline{d}} \Gamma', \overline{\varphi}$*, then* $\vdash_{\overline{d}} \Gamma, \Gamma'$.

**Proof.**

By induction on $\vdash_{\overline{d}} \Gamma, \varphi$ and $\vdash_{\overline{d}} \Gamma', \overline{\varphi}$. □

**Lemma (Degree Reduction)**

$\vdash_{d+1} \subseteq \vdash_{d}$.

*I.e. if* $\vdash_{d+1} \Gamma$ *then* $\vdash_{d} \Gamma$.

**Proof.**

By induction on $\vdash_{d+1}$. For the inductive step:

- By *IH* assume any premisses are already derivable by $\vdash_{d}$.
- Now apply $d$-cut-admissibility. □

We can now complete the proof of cut-elimination:

**Theorem (Cut-Elimination for $\vdash_{d}$, restated)**

*If* $\vdash_{\omega} \Gamma$ *then* $\vdash_{0} \Gamma$.

**Proof.**

We show $\vdash_{d} \subseteq \vdash_{0}$ by induction on $d < \omega$, appealing to *Degree Reduction* at each inductive step. □

As mentioned before, this implies consistency of PA.

# Outline

# Exercises

1. Show, by induction on $\vdash_{\mathsf{PA}}$, that witness term $t$ in the premiss of the $\exists$ rule can be restricted to contain only the variables occurring free in the lower sequent, without loss of provability.
2. Give a $\mathsf{PA}_\omega$ proof of the *Drinkers Paradox*, i.e. show $\vdash_\omega \exists x\, (\varphi(x) \to \forall y\, \varphi(y))$.
3. 
   - For the completeness argument for $\vdash_\omega$, write down the case for an existential formula and a disjunction.
   - What can you conclude about the size of sequents required to carry out the completeness argument?
   - **(Subtle.)** Why might we not want to impose such a restriction for the *Simulation Theorem*, embedding $\vdash_{\mathsf{PA}}$ into $\vdash_\omega$?
4. **(Hard.)** What do you think is the *closure ordinal* of, say, $\vdash_0$ (if you know what that is)?
5. For the simulation of $\vdash_{\mathsf{PA}}$ by $\vdash_\omega$, write down the cases of the missing rules id, $=$, $\neq$ and the ones for $\mathsf{Q}$.
6. 
   - What are the cut-reduction cases for $=$-$\neq$?
   - Write down the $\vee$-$\wedge$ cut-reduction case.

1. A straightforward induction on $\vdash_{\mathsf{PA}}$ yields that, if $\vdash_{\mathsf{PA}} \Gamma(\vec{y})$, for some free variables $\vec{y}$ indicated, then also $\vdash_{\mathsf{PA}} \Gamma(\vec{n})$, for any $\vec{n} \in \mathbb{N}$. From here the statement follows again by induction on $\vdash_{\mathsf{PA}}$, replacing any extraneous free variables in $t$ under the first property.

2. 

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{id}}{\varphi(0) \vdash_\omega \varphi(0)} \quad \dfrac{\dfrac{\dfrac{\text{id}}{\varphi(1) \vdash_\omega \varphi(1)}}{\vdash_\omega \varphi(1),\, \varphi(1) \to \forall y\, \varphi(y)}\,\vee}{\vdash_\omega \varphi(1),\, \exists x\, (\varphi(x) \to \forall y\, \varphi(y))}\,\exists \quad \dfrac{\dfrac{\dfrac{\text{id}}{\varphi(2) \vdash_\omega \varphi(2)}}{\vdash_\omega \varphi(2),\, \varphi(2) \to \forall y\, \varphi(y)}\,\vee}{\vdash_\omega \varphi(2),\, \exists x\, (\varphi(x) \to \forall y\, \varphi(y))}\,\exists \quad \cdots}{\varphi(0) \vdash_\omega \forall y,\, \varphi(y),\, \exists x\, (\varphi(x) \to \forall y\, \varphi(y))}\,\omega}{\vdash_\omega \varphi(0) \to \forall y\, \varphi(y),\, \exists x\, (\varphi(x) \to \forall y\, \varphi(y))}\,\vee}{\vdash_\omega \exists x\, (\varphi(x) \to \forall y\, \varphi(y))}\,\exists$$

# Answers to exercises II

3.

$$
\dfrac{\dfrac{\overset{IH}{\vdash_\omega \varphi(n)}}{\vdash_\omega \exists x\,\varphi(x), \varphi(n)}\,\mathsf{w}}{\vdash_\omega \exists x\,\varphi(x)}\,\exists
\qquad
\dfrac{\dfrac{\overset{IH}{\vdash_\omega \varphi_i}}{\vdash_\omega \varphi_0 \vee \varphi_1, \varphi_i}\,\mathsf{w}}{\vdash_\omega \varphi_0 \vee \varphi_1}\,\vee
$$

where appropriate $n \in \mathbb{N}$ and $i \in \{0, 1\}$ is given by assumptions $\mathfrak{N} \models \exists x \varphi(x)$ and $\mathfrak{N} \models \varphi_0 \vee \varphi_1$, respectively.

4.

5. All of these are simulated by the (in)equational initial sequents and weakening, depending on the environment.

6.
- There are no cut-reduction cases for $=$-$\neq$: we cannot have both $m = n$ and $m \neq n$.

- $$\dfrac{\vee\,\dfrac{\vdash_\omega \Gamma, \varphi_0 \vee \varphi_1, \varphi_i}{\vdash_\omega \Gamma, \varphi_0 \vee \varphi_1} \qquad \wedge\,\dfrac{\vdash_\omega \Gamma', \overline{\varphi}_0 \qquad \vdash_\omega \Gamma', \overline{\varphi}_1}{\vdash_\omega \Gamma', \overline{\varphi}_0 \wedge \overline{\varphi}_1}}{\vdash_\omega \Gamma, \Gamma'}\,\textbf{cut}$$

$$\rightsquigarrow \quad \dfrac{\textbf{cut}\,\dfrac{\vdash_\omega \Gamma, \varphi_0 \vee \varphi_1, \varphi_i \qquad \wedge\,\dfrac{\vdash_\omega \Gamma', \overline{\varphi}_0 \qquad \vdash_\omega \Gamma', \overline{\varphi}_1}{\vdash_\omega \Gamma', \overline{\varphi}_0 \wedge \overline{\varphi}_1}}{\vdash_\omega \Gamma, \Gamma', \varphi_i} \qquad \vdash_\omega \Gamma', \overline{\varphi}_i}{\vdash_\omega \Gamma, \Gamma'}\,\mathit{cut}$$

# Outline

1. A new system: up to $\omega$!
2. Simulating PA
3. Cut-elimination for $\vdash_{\vec{d}}$
4. Break: exercises and questions
5. Formalisation and ordinal analysis

# Taming the metatheory via ordinals

The metatheory required to show completeness of $\vdash_\omega$ for $\mathfrak{N}$ was ridiculously strong.

However, we can tame the metatheory required for the *Simulation Theorem* and the *Cut-Elimination Theorem*, by refining our inductions according to explicit order types.

## Definition

$\varepsilon_0$ is the least ordinal $\alpha$ such that $\omega^\alpha = \alpha$.

## Some observations

- $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\} = \inf\{\alpha : \omega^\alpha \leqslant \alpha\}$ is a limit ordinal.
- Thus $\varepsilon_0$ is countable, and even recursive.
- We can naturally represent and compare ordinals $< \varepsilon_0$ in arithmetic. (More on this tomorrow).

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# Height of judgements

The **height** of a well-founded tree $T$ is defined by induction on $T$ as follows:

- If $T$ has immediate subtrees $\{T_i\}_{i \in I}$, then $\mathrm{ht}(T) := \sup_{i \in I}(\mathrm{ht}(T_i) + 1)$.

So let us write $\vdash^{\alpha}_{\vec{d}} \Gamma$ if there is a derivation of $\vdash_{\vec{d}} \Gamma$ of height $\leqslant \alpha$.

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### Definition

The judgements $\vdash^{\alpha}_{d} \Gamma$, for $d < \omega$ and $\alpha$ an ordinal, are defined by:

$$= \frac{}{\vdash^{0}_{0} n = n} \qquad \neq \frac{}{\vdash^{0}_{0} m \neq n}\ m \neq n$$

$$\mathsf{w}\ \frac{\vdash^{\beta}_{d} \Gamma}{\vdash^{\alpha}_{d} \Gamma, \Gamma'}\ \beta < \alpha \qquad <\!d\text{-}\mathsf{cut}\ \frac{\vdash^{\beta}_{d} \Gamma, \varphi \qquad \vdash^{\gamma}_{d} \Gamma', \overline{\varphi}}{\vdash^{\alpha}_{d} \Gamma, \Gamma'}\ \beta, \gamma < \alpha$$

$$\vee\ \frac{\vdash^{\beta}_{d} \Gamma, \varphi_0 \vee \varphi_1, \varphi_i}{\vdash^{\alpha}_{d} \Gamma, \varphi_0 \vee \varphi_1}\ \beta < \alpha \qquad \wedge\ \frac{\vdash^{\beta}_{d} \Gamma, \varphi \qquad \vdash^{\gamma}_{d} \Gamma, \psi}{\vdash^{\alpha}_{d} \Gamma, \varphi \wedge \psi}\ \beta, \gamma < \alpha$$

$$\exists\ \frac{\vdash^{\beta}_{d} \Gamma, \exists x \varphi(x), \varphi(n)}{\vdash^{\alpha}_{d} \Gamma, \exists x \varphi(x)}\ \beta < \alpha \qquad \omega\ \frac{\vdash^{\beta_0}_{d} \Gamma, \varphi(0) \qquad \vdash^{\beta_1}_{d} \Gamma, \varphi(1) \qquad \vdash^{\beta_2}_{d} \Gamma, \varphi(2) \qquad \cdots}{\vdash^{\alpha}_{d} \Gamma, \forall x \varphi(x)}\ \beta_i < \alpha$$

### Theorem (Simulation, refined)

*If* $\vdash_{\mathsf{PA}} \varphi$*, for* $\varphi$ *closed, then* $\vdash_d^\alpha \varphi$ *for some* $d < \omega$ *and* $\alpha < \omega 2$*.*

**NB:** for formalisation in PRA, we are implicitly using an **evaluator** for $\mathcal{L}_A$-terms.

### Lemma ($d$-cut-admissibility, refined)

*Let* $\deg(\varphi) = d$*. If* $\vdash_d^\alpha \Gamma, \varphi$ *and* $\vdash_d^\beta \Gamma', \overline{\varphi}$*, then* $\vdash_d^{\alpha \sharp \beta} \Gamma, \Gamma'$*.*

**NB:** here $\sharp$ is the **symmetric sum** (or **natural sum**) of ordinals.

### Lemma (Degree reduction)

*If* $\vdash_{d+1}^\alpha \Gamma$ *then* $\vdash_d^{\omega^\alpha} \Gamma$*.*

Putting this all together, we have:

### Theorem (Cut-elimination for PA, refined)

*If* $\mathsf{PA} \vdash \varphi$*, then* $\vdash_0^\alpha \varphi$*, for some* $\alpha < \varepsilon_0$*.*

### Corollary (Consistency, refined)

PA *is consistent, as long as "all* $\alpha < \varepsilon_0$ *are well-founded"…*

# Some remarks on formalisation

The only parts of the refinements of the previous slide *not* available in PRA are the various inductions on ordinals $< \varepsilon_0$.

We can fix some appropriate recursive representation of ordinals $\alpha, \beta, \dots < \varepsilon_0$ and their comparison in $\mathcal{L}_A$ (more on this tomorrow).

Now define $\varepsilon_0$-ind for the schema of axioms for each $\varphi(x)$:

$$\forall \alpha < \varepsilon_0 (\forall \beta < \alpha\, \varphi(\beta) \to \varphi(\alpha)) \to \forall \alpha < \varepsilon_0\, \varphi(\alpha)$$

We can now formalise our entire consistency argument:

**Theorem (Formalised consistency)**

PRA $+\ \varepsilon_0$-ind *proves the consistency of* PA.

**NB.** We have finitistically reduced the entire consistency of PA to a 'simple' computational principle: induction up to $\varepsilon_0$.

**NB.** It is immediate that PA cannot prove well-foundedness of (any appropriate representation of) $\varepsilon_0$.