Proof theory of arithmetic

Lecture 5 – Perspectives and further directions

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These slides are available at http://www.anupamdas.com/esslli23.

Outline

1. Recursion complexity vs induction complexity
2. Higher type recursion
3. Beyond PA and ε_0
4. Conclusions
5. Break: questions and exercises
6. References

Example: Ackermann-Péter, logically

$$A(0,y) = sy$$
$$A(sx, 0) = A(x,1)$$
$$A(sx, sy) = A(x, A(sx,y))$$

Write $a(x,y,z)$ for graph of A. (as a Σ_1-formula)

$PA \vdash \forall x,y \, \exists z \, a(x,y,z)$

Prove $\forall y \exists z \, a(x,y,z)$ by Ind on x

$x=0$: on input y set $z = sy$.

Rank vs logical complexity

It turns out that we can refine our results to account for logical complexity.

Writing $I\Sigma_n$ for the fragment of PA with induction on only Σ_n-formulas:

Theorem

$I\Sigma_n$ has 'proof theoretic ordinal' ω_{n+1}.

Proof idea.

- The restriction of $\vdash_{\mathsf{PA}}$ to $I\Sigma_n$ requires only Σ_n and Π_n formulas, by a partial cut-elimination in PA.
- We only need induction on Σ_r-formulas to prove $<_{r+1}$-wf. □

An excellent reference:

- [Takeuti, 1975]

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The road not taken: higher-type computation

There is another notable successful realisation of Hilbert's Program: the Dialectica functional interpretation.

Here, instead of extending PRA by recursion on higher ordinals, we allow the recursive definition of higher order functionals.

An excellent reference:

- [Avigad and Feferman, 1998]

In recent years there have been astonishing applications via **proof mining**.
An excellent reference:

- [Kohlenbach, 2008]

Finite types

Definition (Finite types)

The **finite types** (or **simple types**), written σ, τ etc., are generated by:

$$\sigma, \tau, \ldots \quad ::= \quad N \quad | \quad \sigma \to \tau$$

Think of types as extra sorts in our logic. We may extend the standard model $\mathfrak{N}$ to a higher **type structure** by setting:

- $N^{\mathfrak{N}} := \mathbb{N}$
- $(\sigma \to \tau)^{\mathfrak{N}} := \{f : \sigma^{\mathfrak{N}} \to \tau^{\mathfrak{N}}\}$

Definition (Informal)

System T extends PRA by appropriate constants and **primitive recursion at all finite types**.

NB: We are being imprecise about higher-type equality here!

Example: Ackermann-Péter, again!

Gödel, 1958

For each formula φ of $\mathcal{L}_A$, there is a quantifier-free T-formula $\varphi_D^N(x, y)$ and a T-term $t(x)$ s.t.:

$$\mathsf{PA} \vdash \varphi \quad \Longrightarrow \quad \mathsf{T} \vdash \varphi_D^N(x, t(x))$$

Gödel, 1958

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This finitistically reduces the consistency of PA to the termination of higher-typed programming language.

Level-by-level refinement

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Gentzen in 1938

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"Indeed, it seems not entirely unreasonable to me to suppose that contradictions might possibly be concealed even in classical analysis. . . . the most important [consistency] proof of all in practice, that for analysis, is still outstanding."

85 YEARS LATER: this is still the *biggest open problem* in proof theory!

Analysis: second-order Peano Arithmetic

The language $\mathcal{L}_2$ of PA2 admits a sort not only for $\mathbb{N}$ $(x, y, \ldots)$, but also type 1 objects (sets, functions, etc.) $(X, Y, \ldots)$.

Crucially we have a **comprehension** axiom,

$$\exists X \, \forall x \, (X(x) \leftrightarrow \varphi(x))$$

for each formula $\varphi(x)$.

We can think of X as:

- A defined predicate.
- A set of natural numbers, with $X(x)$ meaning $x \in X$.
- A real number, with $X(x)$ being the x^{th} bit of X.

Reduction to pure logic

Unlike PA, we can reduce PA2 to pure second-order logic:

$$N(x) \iff \forall X(X(0) \wedge \forall y(X(y) \to X(\mathsf{s}y)) \to X(x))$$

This gives us a definition of $\mathbb{N}$ whence we recover the induction principle.

This reduces Gentzen's problem, consistency of PA2 to cut-elimination for second-order logic...

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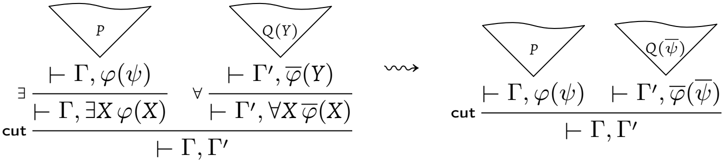
...but there is no free lunch:



Takeuti's conjecture, 1953

Does cut-elimination hold for second-order logic?

Enter Tait, Takahashi and Prawitz

Building on foundational work of Schütte:

Theorem (Tait '66)
Cut is admissible for second-order logic.

Theorem (Takahashi '67, Prawitz '68)
Cut is admissible in Church's simple type theory, i.e. at all finite types.

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However, these arguments are somewhat unsatisfactory:

- They are *non-explicit*: we have no proof-theoretic ordinal for PA2.
- They are *admissibility* results: there is no computational process.

Theorem (Spector '62)

PA2 *is ND-interpreted into an extension of* T *by* ***bar recursion***.

Theorem (Girard '71)

PA2 *is ND-interpreted into a* ***second-order*** *extension of* T.

The story since Gentzen: the first steps into impredicativity

There has nonetheless been significant progress since the days of Gentzen:

Takeuti '67
Ordinal analysis of theories of Π^1_1-comprehension.

Rathjen '90s-'00s
Ordinal analysis of theories of Π^1_2-comprehension.

An excellent reference:

- [Rathjen and Sieg, 2022]

Another inmate on Gentzen, 1945

"He once confided in me that he was really quite content since now he at last had time to think about a consistency proof for analysis. He was in fact fully convinced that he would succeed in carrying out such a proof."

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Conclusions

There are *many* other legacies of proof theory we have not discussed:[1]

- The application of proof theory to complexity theory.
- The application of proof theory to set theory.
- The application of proof theory to type theory and constructive mathematics.

These all constitute highly active areas of research, that are certainly beyond the scope of this course![2]

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[2] …but you can ask me for references.

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Exercises

1. We can think of functions $\mathbb{N} \to \mathbb{N}$ as *streams* (i.e. infinite sequences) of natural numbers. Use (higher-order) primitive recursion to define functionals:
   - $\texttt{hd} : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ returning the first element of a stream input.
   - $\texttt{tl} : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$ by $(n_i)_{i \geqslant 0} \mapsto (n_i)_{i \geqslant 1}$.
2. Show that the two recursive definitions of the Ackermann-Péter function, A (by higher-ordinal recursion) and A' (by higher-type recursion) coincide.
3. Show that second-order logic with equality can be interpreted in second-order logic *without* equality.
4. A function f on the ordinals is **normal** if:
   - (*Strict monotonicity*.) $\alpha < \beta \implies f(\alpha) < f(\beta)$.
   - (*Continuity*.) λ is a limit ordinal $\implies f(\lambda) = \sup_{\alpha < \lambda} (f(\alpha))$.

   Show that:
   - a. f commutes with suprema, i.e. $f(\sup A) = \sup_{\alpha \in A} f(\alpha)$.
   - b. $f(\alpha) \geqslant \alpha$ for all ordinals α.
   - c. f has arbitrarily high fixed points, i.e. for all α there is $\gamma \geqslant \alpha$ with $f(\gamma) = \gamma$.
5. **(Long.)** Show that each primitive recursive function $f(\vec{x})$ is dominated by $A(m, \max(\vec{x}))$, for some $m \in \mathbb{N}$. Conclude that A is not primitive recursive.
6. Write $I\Sigma_1$ and $I\Pi_1$ for the fragments of PA with induction on only Σ_1-formulas and Π_1-formulas, respectively. Show that $I\Sigma_1 = I\Pi_1$.

   (Hard.) Show that $I\Sigma_1$ is even equal to the fragment of PA with induction only on *Boolean combinations* of Σ_1-formulas.

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